MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

AFWL-TR-83-103

AFWL-TR-
83-103

②

AD A139757

BOUNDING OF SIGNAL LEVELS AT TERMINATIONS OF A MULTICONDUCTOR TRANSMISSION-LINE NETWORK

A.K. Agrawal

Dikewood, Div of Kaman Sciences Corp
1613 University Blvd NE
Albuquerque, NM 87102

February 1984

Final Report

Approved for public release; distribution unlimited.

DTIC FILE COPY

DTIC
ELECTE
APR 0 5 1984

E

AIR FORCE WEAPONS LABORATORY
Air Force Systems Command
Kirtland Air Force Base, NM 87117

This final report was prepared by Dikewood, Division of Kaman Sciences Corporation, Albuquerque, New Mexico, under Contract F29601-82-C-0027, Job Order 37630131 with the Air Force Weapons Laboratory, Kirtland Air Force Base, New Mexico. Lt Dennis J. Andersh (NTAA) was the Laboratory Project Officer-in-Charge.

When Government drawings, specifications, or other data are used for any purpose other than in connection with a definitely Government-related procurement, the United States Government incurs no responsibility or any obligation whatsoever. The fact that the Government may have formulated or in any way supplied the said drawings, specifications, or other data, is not to be regarded by implication, or otherwise in any manner construed, as licensing the holder, or any other person or corporation; or as conveying any rights or permission to manufacture, use, or sell any patented invention that may in any way be related thereto.

This report has been authored by a contractor of the United States Government. Accordingly, the United States Government retains a nonexclusive, royalty-free license to publish or reproduce the material contained herein, or allow others to do so, for the United States Government purposes.
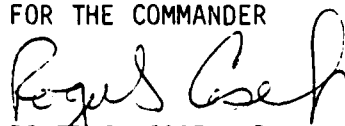
This report has been reviewed by the Public Affairs Office and is releasable to the National Technical Information Services (NTIS). At NTIS, it will be available to the general public, including foreign nations.

If your address has changed, if you wish to be removed from our mailing list, or if your organization no longer employs the addressee, please notify AFWL/NTAA, Kirtland AFB, NM 87117 to help us maintain a current mailing list.

This technical report has been reviewed and is approved for publication.

DENNIS J. ANDERSH
Lt, USAF
Project Officer

FOR THE COMMANDER

DAVID W. GARRISON
Lt Col, USAF
Chief, Applications Branch

ROGER S. CASE, JR
Lt Col, USAF
Chief, Aircraft & Missile Division

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION	1b. RESTRICTIVE MARKINGS
Unclassified	

2a. SECURITY CLASSIFICATION AUTHORITY	3. DISTRIBUTION/AVAILABILITY OF REPORT
	Approved for public release; distribution unlimited.
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE	

4. PERFORMING ORGANIZATION REPORT NUMBER(S)	5. MONITORING ORGANIZATION REPORT NUMBER(S)
DC-FR-1026.230-1	AFWL-TR-83-103

6a. NAME OF PERFORMING ORGANIZATION	6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION
Dikewood, Div of Kaman Sciences Corporation		Air Force Weapons Laboratory (NTAA)

6c. ADDRESS (City, State and ZIP Code)	7b. ADDRESS (City, State and ZIP Code)
1613 University Blvd NE Albuquerque, NM 87102	Kirtland AFB NM 87117

8a. NAME OF FUNDING/SPONSORING ORGANIZATION	8b. OFFICE SYMBOL (If applicable)	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER
		F29601-82-C-0027

8c. ADDRESS (City, State and ZIP Code)	10. SOURCE OF FUNDING NOS.			
	PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT NO.
	64711F	3763	01	31

11. TITLE (Include Security Classification) BOUNDING OF SIGNAL LEVELS AT TERMINATIONS OF A MULTICONDUCTOR TRANSMISSION-LINE NETWORK (U)

12. PERSONAL AUTHOR(S)
Agrawal, Ashok K.

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Yr., Mo., Day)	15. PAGE COUNT
Final Report	FROM Jun 82 TO Apr 83	1984, February	155

16. SUPPLEMENTARY NOTATION

17.	COSATI CODES		18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB. GR.	Coupling Multiconductor Transmission Lines
09	01		EMP Signal Bounds
			Norms Transmission-Line Networks

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

Starting from the norm concept for vectors and matrices, this report addresses the problem of bounding signal levels at terminations of a multiconductor transmission-line network. The overall network equation is formulated in terms of the combined voltage supervector (a special combination of the voltage and current vectors). Utilizing the scattering and propagation supermatrices for the waves on the transmission-line network and the combined voltage supervector for sources, the BLT equation is used to express the combined voltage supervectors and the voltage and current supervectors at the junctions. The upper and lower bounds for the combined voltage supervector, voltage supervector and current supervector are obtained in terms of the norms of the propagation and scattering supermatrices and the norm of the combined voltage source supervector. Various properties of the propagation and scattering supermatrices are discussed for two cases, namely, a uniform section of a multiconductor transmission line and a multiconductor transmission line with a branch.

(over)

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT	21. ABSTRACT SECURITY CLASSIFICATION
UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT. ☐ OTIC USERS ☐	Unclassified

22a. NAME OF RESPONSIBLE INDIVIDUAL	22b. TELEPHONE NUMBER (Include Area Code)	22c. OFFICE SYMBOL
Lt Dennis J. Andersh	(505) 844-0326	AFWL/NTAA

DD FORM 1473, 83 APR   EDITION OF 1 JAN 73 IS OBSOLETE.

19. ABSTRACT. (Cont)

The experssions for upper and lower bounds for combined voltage supervectors and voltage and current supervectors are derived. Various norms of vectors, matrices, supervectors, and supermatrices are also discussed.
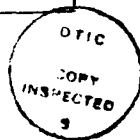
PREFACE

The author would like to thank Dr. Carl E. Baum for many illuminating discussions during the course of this work, and Mr. Terry Brown for his help in computations.

CONTENTS

3

CONTENTS (Concluded)

I. INTRODUCTION

Designing or analyzing the response of an electronic system to some kind of electromagnetic interference such as the nuclear electromagnetic pulse (EMP) is a complex problem, because of so many individual components and interconnections. An example of such a complex system is the multiconductor cable network inside an aircraft.

When an aircraft is in an EMP environment, the cables inside the aircraft will be excited by an electromagnetic field which penetrates the aircraft body through a large number of penetrating conductors, small antennas, apertures, and by diffusion through the skin of the aircraft. Many parameters determine the response of a multiconductor cable; these include polarization, angle of incidence, planarity and spectral contents of the incident field; number of points of entry (POEs); size, shape and location of POEs; physical properties of the transmission line and the surrounding medium; and the configuration of load impedances. This large number of variables and the complexity of the multiconductor cables create problems in gaining insight into how to control the system performance in an electromagnetic environment.

In evaluating the system vulnerability to EMP, it is often desirable to evaluate upper bounds on the problem rather than compute the full coupling and interaction evaluation for the cases of interest so as to determine the system survivability/vulnerability with high confidence. In most of the cases the latter may be effectively impossible due to system complexity and lack of complete and correct definition. To deal with this complexity one needs ways to identify and deal with a set of important variables which, if

controlled, control the system performance. An approach to this problem has been developed (Ref. 1) which can be referred to as electromagnetic topology.

Having defined the electromagnetic topology and the related interaction sequence diagram (graph), one can write a general matrix equation (BLT equation, Ref. 1). The resulting supermatrix equation evaluates into an approximate solution which shows the dependence of the system performance on system shielding parameters. A BLT equation for transmission-line networks within the system can also be formulated (Ref. 2). This equation shows the dependence of the cable network response on the induced sources, physical configuration of the cables in the network, and the load configurations. Certain approximate bounds for the termination voltages and currents can be obtained from norm concepts (Ref. 3).

In Reference 4, upper bounds were obtained for voltages and currents at terminations of a multiconductor transmission line excited by a single aperture, but bounds were not established for physical parameters of the line. For a moderately mismatched termination, the upper bound for the termination voltage was 10 times the actual maximum voltage.

This report establishes upper and lower bounds on the voltages and currents at terminations of a multiconductor transmission-line network excited by an external electromagnetic field. The general matrix equation (BLT equation) is used as the basis for establishing upper and lower bounds on the termination voltages and currents. Upper and lower bounds on forward and backward traveling combined voltage waves are also established. These bounds are obtained in terms of upper bounds of several parameters, such as the source, load impedances, characteristic impedance of the line, etc. Upper bounds on these parameters are established for some special cases.

In Section II, the equations governing the response of a general multi-conductor transmission-line network are discussed. In Section III, the upper and lower bounds for the combined voltages, voltages, and currents are obtained in terms of the induced sources, physical properties of the cable network, and the load configurations. The bounds on the ratio of the maximum pin current to the bundle current are also discussed. In Sections IV and V, bounds are obtained for two special cases of a general multiconductor transmission-line network, namely, a uniform section of a multiconductor transmission line and a multiconductor transmission line with a branch. Procedures for obtaining bounds on parameters of the line such as the characteristic-impedance matrix, reflection-coefficient matrix, and the scattering matrix are discussed. Bounds for induced sources are also discussed for these two cases. In Section VI, the upper and lower bounds are computed for a uniform three-conductor line terminated at both ends in diagonal loads for several load configurations. These bounds are compared with the calculated maximum values using the QV7TA computer code (Ref. 5).

II. GENERAL MULTICONDUCTOR TRANSMISSION-LINE NETWORK EQUATIONS

This section reviews the multiconductor transmission-line equations for a general network. The detailed derivation of these equations is discussed in Reference 2. These equations form a basis for the evaluation of upper bounds on voltages and currents at terminations of a multiconductor line network.

1. PROPAGATION ON A UNIFORM N-WIRE TRANSMISSION LINE

First consider a single section of an N-wire transmission line. An N-wire transmission line is one that consists of N conductors and a reference conductor (or an equivalent one). Figure 1 shows a per-unit-length equivalent circuit of the line with distributed sources. The equations governing the voltage and current propagation on an N-wire transmission line are the generalized multiconductor transmission-line equations:

$$\frac{d}{dz} (\tilde{I}_n(z,s)) = -(\tilde{Y}'_{n,m}(s)) \cdot (\tilde{V}_n(z,s)) + (\tilde{I}_n^{(s)'}(z,s)) \qquad (1)$$

$$\frac{d}{dz} (\tilde{V}_n(z.s)) = -(\tilde{Z}'_{n,m}(s)) \cdot (\tilde{I}_n(z,s)) + (\tilde{V}_n^{(s)'}(z,s)) \qquad (2)$$

where

$$s = j\omega$$

z = position along the line

$(\tilde{I}_n(z,s))$ = current vector at z

$(\tilde{V}_n(z,s))$ = voltage vector at z

$(\tilde{Y}'_{n,m}(s))$ = per-unit-length shunt admittance matrix

$(\tilde{Z}'_{n,m}(s))$ = per-unit-length series impedance matrix
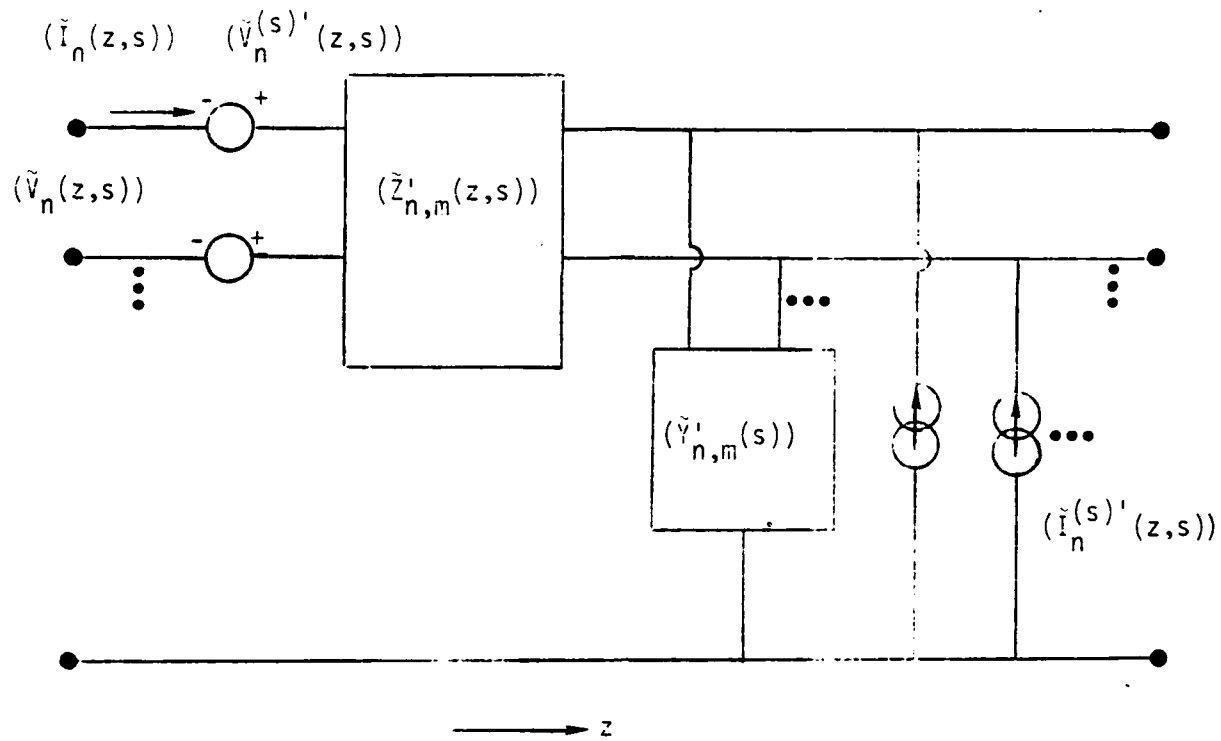
Figure 1. The per-unit-length model of a multiconductor transmission line.

$$(\tilde{I}_n^{(s)\,'}(z,s)) = \text{per-unit-length shunt current source vector}$$

$$(\tilde{V}_n^{(s)\,'}(z,s)) = \text{per-unit-length series voltage source vector}$$

Note that all vectors are of dimension N, and all matrices are N x N.

By algebraic manipulations of Equations 1 and 2, an equation for combined voltages can be obtained as follows (Ref. 2):

$$\left[(1_{n,m}) \frac{d}{dz} + q(\tilde{\gamma}_{c_{n,m}}(s))\right] \cdot (\tilde{V}_n(z,s))_q = (\tilde{V}_n^{(s)\,'}(z,s))_q \qquad (3)$$

$$1_{n,m} = \begin{cases} 1 \text{ for } n = m \\ 0 \text{ for } n \neq m \end{cases} \qquad (4)$$

q = ± for forward and backward traveling combined N-vector
waves, respectively

9

$$(\tilde{\gamma}_{c_{n,m}}(s)) = \left[(\tilde{Z}'_{n,m}(s)) \cdot (\tilde{Y}'_{n,m}(s))\right]^{\frac{1}{2}} \qquad (5)$$

$$(\tilde{V}_n(z,s))_q = (\tilde{V}_n(z,s)) + q(\tilde{Z}_{c_{n,m}}(s)) \cdot (\tilde{I}_n(z,s))$$

$$(\tilde{V}_n^{(s)'}(z,s))_q = (\tilde{V}_n^{(s)'}(z,s)) + q(\tilde{Z}_{c_{n,m}}(s)) \cdot (\tilde{I}_n^{(s)'}(z,s)) \qquad (6)$$

$$(\tilde{Z}_{c_{n,m}}(s)) = (\tilde{\gamma}_{c_{n,m}}(s))^{-1} \cdot (\tilde{Z}'_{n,m}(s)) \qquad (7)$$

$$(\tilde{Y}_{c_{n,m}}(s)) = (\tilde{Z}_{c_{n,m}}(s))^{-1} \qquad (8)$$

$(\tilde{Z}_{c_{n,m}}(s)) \equiv$ characteristic-impedance matrix

$(\tilde{Y}_{c_{n,m}}(s)) \equiv$ characteristic-admittance matrix

Substituting $q = +1$ and $q = -1$ in Equation 6, the following relations can be obtained:

$$(\tilde{V}_n(z,s))_+ = (\tilde{V}_n(z,s)) + (\tilde{Z}_{c_{n,m}}(s)) \cdot (\tilde{I}_n(z,s)) \qquad (9)$$

$$(\tilde{V}_n(z,s))_- = (\tilde{V}_n(z,s)) - (\tilde{Z}_{c_{n,m}}(s)) \cdot (\tilde{I}_n(z,s)) \qquad (10)$$

$$(\tilde{V}_n^{(s)'}(z,s))_+ = (\tilde{V}_n^{(s)'}(z,s)) + (\tilde{Z}_{c_{n,m}}(s)) \cdot (\tilde{I}_n^{(s)}(z,s)) \qquad (11)$$

$$(\tilde{V}_n^{(s)'}(z,s))_- = (\tilde{V}_n^{(s)'}(z,s)) - (\tilde{Z}_{c_{n,m}}(s)) \cdot (\tilde{I}_n^{(s)}(z,s)) \qquad (12)$$

$(\tilde{V}_n(z,s))_+ \equiv$ forward traveling combined voltage vector or wave

$(\tilde{V}_n(z,s))_- \equiv$ backward traveling combined voltage vector or wave

From Equations 9 and 10, voltage and current vectors can be reconstructed in terms of forward and backward waves, and given by the following relations:

$$(\tilde{V}_n(z,s)) = \frac{1}{2}\left[(\tilde{V}_n(z,s))_+ + (\tilde{V}_n(z,s))_-\right] \tag{13}$$

$$(\tilde{I}_n(z,s)) = \frac{1}{2}(\tilde{Y}_{c_{n,m}}(s)) \cdot \left[(\tilde{V}_n(z,s))_+ - (\tilde{V}_n(z,s))_-\right] \tag{14}$$

From the above definitions can be obtained two sets of waves propagating in opposite directions along z. For all modes, there are

$$\exp\left[-(\tilde{\gamma}_{c_{n,m}}(s))z\right] \quad + \text{ propagating}$$

$$\exp\left[(\tilde{\gamma}_{c_{n,m}}(s))z\right] \quad - \text{ propagating}$$

Equation 3 can be integrated to obtain a solution for the combined voltage vectors to give

$$(\tilde{V}_n(z,s))_q = \exp\left\{-q(\tilde{\gamma}_{c_{n,m}}(s))[z - z_0]\right\} \cdot (\tilde{V}_n(z_0,s))_q$$

$$+ \int_{z_0}^{z} \exp\left\{-q(\gamma_{c_{n,m}}(s))[z - z']\right\} \cdot (\tilde{V}_n^{(s)'}(z',s))_q dz' \tag{15}$$

For a + wave (i.e., a wave propagating in the +z direction), assume that $(\tilde{V}_n(0,s))_+$ is specified, then Equation 15 gives

$$(\tilde{V}_n(z,s))_+ = \exp\left\{-(\tilde{\gamma}_{c_{n,m}}(s))z\right\} \cdot (\tilde{V}_n(0,s))_+$$

$$+ \int_0^{z} \exp\left\{-(\tilde{\gamma}_{c_{n,m}}(s))[z - z']\right\} \cdot (\tilde{V}_n^{(s)'}(z',s))_+ dz' \tag{16}$$

Similarly for a - wave with $(\tilde{V}_n(L,s))_-$ assumed specified, we have

$$(\tilde{V}_n(z,s))_- = \exp\left\{(\tilde{\gamma}_{c_{n,m}}(s))[z - L]\right\} \cdot (\tilde{V}_n(L,s))_-$$

$$+ \int_L^z \exp\left\{(\tilde{\gamma}_{c_{n,m}}(s))[z - z']\right\} \cdot (\tilde{v}_n^{(s)'}(z',s))_- dz' \qquad (17)$$

These results illustrate that the + wave depends only on the left boundary condition and the - wave depends only on the right boundary condition in a very compact way.

2.   TERMINATION CONDITION OF A SINGLE SECTION OF THE LINE (TUBE)

A transmission line is usually terminated at the two ends z = 0 and z = L. The termination could be a lumped impedance, a distributed network, open circuit or short circuit. If sources are included, these conditions can be represented by a generalized Thévenin equivalent network or a generalized Norton equivalent network.

Passive terminations can be specified as an impedance matrix $(\tilde{Z}_{T_{n,m}}(z,s))$ or an admittance matrix $(\tilde{Y}_{T_{n,m}}(z,s))$, where z = 0 or L. The terminating conditions can be specified by scattering matrices $(\tilde{S}_{n,m}(z,s))$, where z = 0 or L. Consider at z = L (see Fig. 2); let the incoming waves be designated by a superscript - and the outgoing waves by a +. The scattering matrix is defined by

$$(\tilde{\mathcal{V}}^{(+)}(s)) = (\tilde{S}_{n,m}(z,s)) \cdot (\tilde{\mathcal{V}}_n^{(-)}(s)) \qquad (18)$$

For the case illustrated in Figure 2, observe that, if this termination is taken as z = L, then
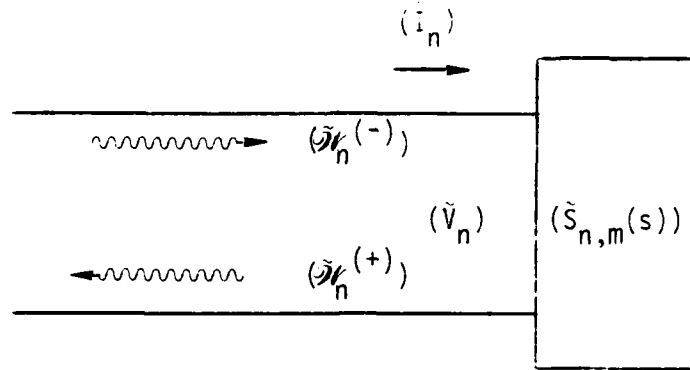
Figure 2. Incoming and outgoing wave at a junction.

$$(\tilde{\mathscr{V}}_n^{(+)}(s)) = (\tilde{V}_n(L,s))_-$$

$$(\tilde{\mathscr{V}}_n^{(-)}(s)) = (\tilde{V}_n(L,s))_+$$
(19)

And if the termination is taken as $z = 0$, then

$$(\tilde{\mathscr{V}}_n^{(+)}(s)) = (\tilde{V}_n(0,s))_+$$

$$(\tilde{\mathscr{V}}_n^{(-)}(s)) = (\tilde{V}_n(0,s))_-$$
(20)

Equation 18 for $z = 0$ and $z = L$ can then be rewritten as

$$(\tilde{V}_n(L,s))_- = (\tilde{S}_{n,m}(L,s)) \cdot (\tilde{V}_n(L,s))_+$$
(21)

$$(\tilde{V}_n(0,s))_+ = (\tilde{S}_{n,m}(0,s)) \cdot (\tilde{V}_n(0,s))_-$$
(22)

which in this terminating case is the same as the definition of a reflection matrix, and these are given by the following relations:

13

$$(\tilde{S}_{n,m}(L,s)) = \left[(\tilde{Z}_{T_{n,m}}(L,s)) + (\tilde{Z}_{c_{n,m}}(s))\right]^{-1} \cdot \left[(\tilde{Z}_{T_{n,m}}(L,s)) - (\tilde{Z}_{c_{n,m}}(s))\right] \quad (23)$$

$$(\tilde{S}_{n,m}(0,s)) = \left[(\tilde{Z}_{T_{n,m}}(0,s)) + (\tilde{Z}_{c_{n,m}}(s))\right]^{-1} \cdot \left[(\tilde{Z}_{T_{n,m}}(0,s)) - (\tilde{Z}_{c_{n,m}}(s))\right] \quad (24)$$

The scattering matrices in Equations 23 and 24 can also be represented in terms of the characteristic-admittance matrix and the load-admittance matrix as

$$(\tilde{S}_{n,m}(L,s)) = \left[(\tilde{Y}_{c_{n,m}}(s)) + (\tilde{Y}_{T_{n,m}}(L,s))\right]^{-1} \cdot \left[(\tilde{Y}_{c_{n,m}}(s)) - (\tilde{Y}_{T_{n,m}}(L,s))\right] \quad (25)$$

$$(\tilde{S}_{n,m}(0,s)) = \left[(\tilde{Y}_{c_{n,m}}(s)) + (\tilde{Y}_{T_{n,m}}(0,s))\right]^{-1} \cdot \left[(\tilde{Y}_{c_{n,m}}(s)) - (\tilde{Y}_{T_{n,m}}(0,s))\right] \quad (26)$$

Having defined the general transmission-line equations and termination conditions for a uniform multiconductor transmission line, now consider multi-tube multiconductor transmission-line networks. Before deriving the BLT equation, the scattering supermatrix for a general network will be discussed.

3. SCATTERING SUPERMATRIX

The concept of scattering matrices introduced in the previous section for a terminated tube is extended here for junctions where more than one tube is connected. Collections and suitable ordering of scattering matrices at all junctions of the transmission-line network form a scattering supermatrix.

a. _Junction scattering supermatrix_--Consider the νth junction, J_{ν}, with tube ends denoted by $J_{\nu;r}$ with index r denoting the rth tube. Let this junction be characterized by an impedance matrix

14

$$(\tilde{Z}_{n,m}(s))_\nu = (\tilde{Y}_{n,m}(s))^{-1} \qquad (27)$$

The junction scattering matrix is defined so that

$$(\tilde{V}_n(s))_{\nu,+} = (\tilde{S}_{n,m}(s))_\nu \cdot (\tilde{V}_n(s))_{\nu,-}$$

where the subscripts + and - refer to the aggregate of respectively outgoing and incoming waves on the various tubes in the form of combined voltage vectors.

In the supermatrix, form partition according to waves on the r_ν tube ends connected to J_ν as

$$((\tilde{V}_n^{(0)}(s))_r)_\nu = (\tilde{Z}_{n,m}(s)_{r,r'})_\nu : (\tilde{I}_n^{(0)}(s))_\nu$$

$$\qquad (28)$$

$$((\tilde{Y}_{n,m}(s))_{r,r'})_\nu \equiv ((\tilde{Z}_{n,m}(s))_{r,r'})_\nu^{-1}$$

where

$$(\tilde{V}_n^{(0)}(s))_{r;\nu} \quad , \quad (\tilde{I}_n^{(0)}(s))_{r;\nu}$$

$$r = 1,2,\ldots,r_\nu$$

are the voltage and current vectors on the rth tube ends at J_ν with current convention into J_ν.

The tube associated with the rth tube end at J_ν has characteristic impedance and admittance matrices which can be put in supermatrix form for J_ν as

$$((\tilde{Z}_{c_{n,m}}(s))_{r,r'})_\nu \equiv \text{tube-end characteristic-impedance supermatrix for } J_\nu$$

$$\qquad (29)$$

$$((\tilde{Y}_{c_{n,m}}(s))_{r,r'})_\nu \equiv \text{tube-end characteristic-admittance supermatrix for } J_\nu$$

15

where

$$(\tilde{Z}_{c_{n,m}}(s))_{r,r';\nu} \equiv \begin{cases} \text{characteristic-impedance matrix for } r\text{th} \\ \text{tube end at } J_\nu \text{ for } r = r' \\ \\ (0_{n,m}) \text{ for } r \neq r' \end{cases}$$

$$(\tilde{Y}_{c_{n,m}}(s))_{r,r';\nu} \equiv \begin{cases} \text{characteristic-admittance matrix for } r\text{th} \\ \text{tube end at } J_\nu \text{ for } r = r' \\ \\ (0_{n,m}) \text{ for } r \neq r' \end{cases}$$

$$\tag{30}$$

$$(\tilde{Y}_{c_{n,m}}(s))_{r,r;\nu} = (\tilde{Z}_{c_{n,m}}(s))^{-1}_{r,r;\nu}$$

The impedance and admittance supermatrices for the tube ends at a given junction are block diagonal and may be represented in terms of the direct sum $\oplus$ as

$$((\tilde{Z}_{c_{n,m}}(s))_{r,r'})_\nu \equiv (\tilde{Z}_{c_{n,m}}(s))_{1,1;\nu} \oplus (\tilde{Z}_{c_{n,m}}(s))_{2,2;\nu} \oplus \cdots \oplus (\tilde{Z}_{c_{n,m}}(s))_{r_\nu,r_\nu;\nu}$$

$$\equiv \bigoplus_{\nu=1}^{r_\nu} (Z_{c_{n,m}}(s))_{r,r;\nu}$$

$$((\tilde{Y}_{c_{n,m}}(s))_{r,r'})_\nu \equiv (\tilde{Y}_{c_{n,m}}(s))_{1,1;\nu} \oplus (\tilde{Y}_{c_{n,m}}(s))_{2,2;\nu} \oplus \cdots \oplus (\tilde{Y}_{c_{n,m}}(s))_{r_\nu,r_\nu;\nu}$$

$$\equiv \bigoplus_{\nu=1}^{r_\nu} (\tilde{Y}_{c_{n,m}}(s))_{r,r;\nu} \tag{31}$$

The scattering supermatrix for J_ν is defined by

$$((\tilde{V}_n(s))_r)_{\nu,+} \equiv ((\tilde{S}_{n,m}(s))_{r,r'})_\nu : ((\tilde{V}_n(s))_r)_{\nu,-}$$

$$((\tilde{V}_n(s))_r)_{\nu,+} \equiv ((\tilde{V}_n^{(0)}(s))_r)_\nu - ((\tilde{Z}_{c_{n,m}}(s))_{r,r'})_\nu : ((\tilde{I}_n^{(0)}(s))_r)_\nu$$

$$\equiv \text{outgoing wave supervector at } J_\nu$$

$$((\tilde{V}_n(s))_r)_{\nu,-} \equiv ((\tilde{V}_n^{(0)}(s))_r)_\nu + ((\tilde{Z}_{c_{n,m}}(s))_{r,r'})_\nu : ((\tilde{I}_n^{(0)}(s))_r)_\nu$$

$$\equiv \text{incoming wave supervector at } J_\nu \tag{32}$$

16

By solving Equations 28 and 32, the junction scattering supermatrix can be obtained as (Ref 2):

$$((\breve{S}_{n,m}(s))_{r,r'})_{\nu} = \left[((\tilde{Z}_{n,m}(s))_{r,r'})_{\nu} : ((\tilde{Y}_{c_{n,m}}(s))_{r,r'})_{\nu} + ((1_{n,m})_{r,r'})_{\nu} \right]^{-1}$$

$$: \left[((\tilde{Z}_{n,m}(s))_{r,r'})_{\nu} : ((\tilde{Y}_{c_{n,m}}(s))_{r,r'})_{\nu} - ((1_{n,m})_{r,r'})_{\nu} \right]$$

$$= \left[((1_{n,m})_{r,r'})_{\nu} + ((\tilde{Z}_{c_{n,m}}(s))_{r,r'})_{\nu} : ((\tilde{Y}_{n,m}(s))_{r,r'})_{\nu} \right]^{-1}$$

$$: \left[((1_{n,m})_{r,r'})_{\nu} - ((\tilde{Z}_{c_{n,m}}(s))_{r,r'})_{\nu} : ((\tilde{Y}_{n,m}(s))_{r,r'})_{\nu} \right]$$

(33)

b. <u>Scattering supermatrix</u>--The proper ordering of all the junction scattering matrices into one large matrix forms the system (or network) scattering supermatrix $((\tilde{S}_{n,m}(s))_{u,v})$. This supermatrix is a collection of the junction scattering matrices, which themselves are collections of individual tube scattering matrices. The latter are matrices containing reflection and transmission coefficients of individual wires within the tubes.

The wave-wave matrix $(W_{u,v})$ gives the structure of the scattering supermatrix since the scattering supermatrix is in general block sparse as

$$((\tilde{S}_{n,m}(s))_{u,v}) = ((0_{n,m})_{u,v}) \quad \text{for } W_{u,v} = 0 \tag{34}$$

The network elementary scattering matrices are formed as

$$(\tilde{S}_{n,m}(s))_{u,v} \equiv \begin{cases} (\tilde{S}_{n,m}(s))_{r,r';\nu} & \text{for } \nu_1 = \nu_2 = \nu \text{ or } W_v \\ & \text{scattering into } W_u \text{ at } J_\nu \\ (0_{n,m}) = (0_{n,m})_{u,v} & \text{for } \nu_1 \neq \nu_2 \text{ or } W_v \text{ not} \\ & \text{scattering into } W_u \end{cases}$$

(35)

17

The wave-wave matrix is defined as

$$
W_{u,v} = \begin{cases} 1 \text{ for } \nu_1 = \nu_2 = \nu \text{ and } W_v \text{ scattering into } W_u \text{ at } J_\nu \\ \\ 0 \text{ for } \nu_1 \neq \nu_2 \text{ or } W_v \text{ not scattering into } W_u \end{cases}
$$
(36)

The scattering supermatrix is $N_W \times N_W$ in terms of the u,v indices, i.e.,

$$
u,v = 1,2,\ldots,N_W
$$
(37)

where N_W is equal to twice the number of tubes. The elementary scattering matrices $(\tilde{S}_{n,m}(s))_{u,v}$ are $N_u \times N_v$, i.e.,

$$
\begin{aligned} n &= 1,2,\ldots,N_u \\ m &= 1,2,\ldots,N_v \end{aligned}
$$
(38)

where

N_u = number of conductors (not including reference) on the (39)
tube with uth wave

and likewise for N_v.

As a special case, if there are no selftubes (with both ends connected to the same junction), then

$$
W_{u,u} = 0 \quad \text{for } u = 1,2,\ldots,N_W \text{ for no selftubes}
$$

$$
(\tilde{S}_{n,m}(s))_{u,u} = (0_{n,m})_{u,u} \quad \text{for } n,m = 1,2,\ldots,N_u \text{ (square)}
$$
(40)

4. DEFINITIONS OF SEVERAL IMPORTANT SUPERMATRIX AND SUPERVECTOR QUANTITIES

This section takes the results for the combined voltages on a tube and separates them into wave variables for the network. The resulting equation for a general combined voltage wave W_u is used to relate the combined voltage waves at both ends of the tube with the sources along the tube. Each term is

18

generalized to a form appropriate to the transmission-line network, i.e., supermatrices and supervectors, by aggregating the results for all W_u for $u = 1,2,\ldots,N_W$.

First, identify the two waves on the tube with the two waves of the transmission-line network, e.g., W_u and W_v.

Then, consider the + wave; call this W_u and set the coordinate and dimension variable as

$$
\begin{aligned}
&L_u \equiv L \equiv \text{length of path for } W_u \\
&z_u \equiv z \equiv \text{wave coordinate for } W_u \\
&0 \leq z_u \leq L_u \\
&N_u \equiv N \equiv \text{number of conductors (less reference) on tube and} \\
&\qquad\qquad\qquad \text{dimension of vectors for } W_u
\end{aligned}
\tag{41}
$$

The wave and source conventions are then

$$
(\tilde{V}_n(z_u,s))_u \equiv (\tilde{V}_n(z,s))_+ = (\tilde{V}_n(z_u,s)) + (\tilde{Z}_{c_{n,m}}(s))_u \cdot (\tilde{I}_n(z_u,s))
$$

$$
\equiv \text{combined voltage for } W_u
$$

$$
(\tilde{V}_n^{(s)'}(z_u,s))_u \equiv (\tilde{V}_n^{(s)'}(z,s))_+ = (\tilde{V}_n^{(s)'}(z_u,s)) + (\tilde{Z}_{c_{n,m}}(s))_u \cdot (\tilde{I}_n^{(s)'}(z_u,s))
$$

$$
\equiv \text{combined voltage source per unit length for } W_u
$$

$$
(\tilde{Z}_{c_{n,m}}(s))_u \equiv (\tilde{Y}_{c_{n,m}}(s))_u^{-1} = \text{characteristic-impedance matrix for } W_u
$$

$$
(\tilde{Y}_{c_{n,m}}(s))_u \equiv (\tilde{Y}_{c_{n,m}}(s)) \equiv \text{propagation matrix for } W_u \tag{42}
$$

The combined voltage vector for the wave W_u is given by

19

$$\left(\tilde{V}_n(z_u,s)\right)_u = \exp\left\{-\left(\tilde{\gamma}_{c_{n,m}}(s)\right)_u z_u\right\} \cdot \left(\tilde{V}_n(0,s)\right)_u$$

$$+ \int_0^{z_u} \exp\left\{-\left(\tilde{\gamma}_{c_{n,m}}(s)\right)_u \left[z_u - z_u'\right]\right\} \cdot \left(\tilde{v}_n^{(s)'}(z_u',s)\right)_u dz_u'$$

(43)

Similarly, the combined voltage vector for the wave W_v can be defined. Equation 43 shows the combined voltage at any z_u in terms of the value (boundary condition) at $z_u = 0$. Setting $z_u = L_u$ introduces the boundary value there as giving

$$\left(\tilde{V}_n(L_u,s)\right)_u = \exp\left\{-\left(\tilde{\gamma}_{c_{n,m}}(s)\right)_u L_u\right\} \cdot \left(\tilde{V}_n(0,s)\right)_u$$

$$+ \int_0^{L_u} \exp\left\{-\left(\tilde{\gamma}_{c_{n,m}}(s)\right)_u \left[L_u - z_u'\right]\right\} \cdot \left(\tilde{v}_n^{(s)'}(z_u',s)\right)_u dz_u'$$

(44)

This evidently relates $\left(\tilde{V}_n(0,s)\right)_u$ which is an outgoing wave from the junction at $z_u = 0$, to $\left(\tilde{V}_n(L_u,s)\right)_u$ which is an incoming wave to the junction at $z_u = L_u$.

As a matter of convention, let all the sources be considered as being present in the tubes instead of at the junction. If a junction has an equivalent circuit containing sources, then the sources can be moved just across the terminals into the tube, a movement of zero distance.

a. Propagation characteristics supermatrix--Considering the various terms in Equation 44, first aggregate all the propagation terms not associated with the sources into a block diagonal propagation supermatrix as

20

$$((\tilde{\Gamma}_{n,m}(s))_{u,v})$$

$$\equiv \exp\left\{-(\tilde{\gamma}_{c_{n,m}}(s))_1 L_1\right\} \oplus \exp\left\{-(\tilde{\gamma}_{c_{n,m}}(s))_2 L_2\right\} \oplus \cdots \oplus \exp\left\{-(\tilde{\gamma}_{c_{n,m}}(s))_{N_W} L_{N_W}\right\}$$

$$\equiv \overset{N_W}{\underset{u=1}{\oplus}} \exp\left\{-(\tilde{\gamma}_{c_{n,m}}(s))_u L_u\right\} \tag{45}$$

$$\equiv \text{propagation supermatrix}$$

where the elementary matrices (blocks) are given by

$$(\tilde{\Gamma}_{n,m}(s))_{u,v} = \begin{cases} \exp\left\{-(\tilde{\gamma}_{c_{n,m}}(s))_u L_u\right\} & \text{for } u = v \\ (0_{n,m}) & \text{for } u \neq v \end{cases}$$

$$\tag{46}$$

$$= 1_{u,v} \exp\left\{-(\tilde{\gamma}_{c_{n,m}}(s))_u L_u\right\}$$

b. <u>Source supervector and combined voltage supervector</u>--From Equation 44 define a source vector for W_u in traveling from $z_u = 0$ to $z_u = L_u$ as

$$(\tilde{V}_n^{(s)}(s))_u \equiv \int_0^{L_u} \exp\left\{-(\tilde{\gamma}_{c_{n,m}}(s))_u [L_u - z_u']\right\} \cdot (\tilde{V}_n^{(s)'}(z_u',s))_u dz_u' \tag{47}$$

The source supervector is then

$$((\tilde{V}_n^{(s)}(s))_u) = \left(\int_0^{L_u} \exp\left\{-(\tilde{\gamma}_{c_{n,m}}(s))_u [L_u - z_u']\right\} \cdot (\tilde{V}_n^{(s)'}(z_u',s))_u dz_u' \right) \tag{48}$$

For completeness, one has the aggregate of combined voltage vectors in Equation 43 as

$$((\tilde{V}_n(0,s))_u) \equiv \text{combined voltage supervector of outgoing waves at the junctions}$$

$$\tag{49}$$

$$((\tilde{V}_n(L_u,s))_u) \equiv \text{combined voltage supervector of incoming waves at junctions}$$

21

5. BLT EQUATION

Combining the results of the previous derivations we can write the BLT equation for the description of the transmission-line network. In Reference 2 the BLT equation was derived for the combined voltage waves leaving the junctions. Here, the BLT equation will be derived for four variables, namely, combined voltage waves leaving the junctions, combined voltage waves entering the junctions, the total voltage vectors at the junctions, and the total current vectors at the junctions. Begin with the scattering supermatrix which relates the incoming waves to the outgoing waves as

$$(\tilde{V}_n(0,s))_u = ((\tilde{S}_{n,m}(s))_{u,v}) : ((\tilde{V}_n(L_u,s)_u)) \tag{50}$$

Next, relate the incoming waves at the output ends of the tubes $(z_u = L_u)$ to the same waves at the input end of the same tubes $(z_u = 0)$, albeit at different junctions in general. Equation 44 in supermatrix form is

$$((\tilde{V}_n(L_u,s))_u) = ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{V}_n(0,s))_u) + ((\tilde{V}_n^{(s)}(s))_u) \tag{51}$$

Combining Equations 50 and 51 gives

$$((\tilde{V}_n(0,s))_u) = ((\tilde{S}_{n,m}(s))_{u,v}) : ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{V}_n(0,s))_u)$$

$$+ ((\tilde{S}_{n,m}(s))_{u,v}) : ((\tilde{v}_n^{(s)}(s))_u) \tag{52}$$

That is rearranged by use of the supermatrix identity as

$$\left[((1_{n,m})_{u,v}) - ((\tilde{S}_{n,m}(s))_{u,v}) : ((\tilde{\Gamma}_{n,m}(s))_{u,v})\right] : ((\tilde{V}_n(0,s))_u)$$

$$= ((S_{n,m}(s))_{u,v}) : ((\tilde{v}_n^{(s)}(s))_u) \tag{53}$$

22

This can be rearranged to obtain

$$((\tilde{V}_n(0,s))_u) = \left[((1_{n,m})_{u,v}) - ((\tilde{S}_{n,m}(s))_{u,v}) : ((\tilde{\Gamma}_{n,m}(s))_{u,v}\right]^{-1}$$

$$: ((\tilde{S}_{n,m}(s))_{u,v}) : ((\tilde{V}_n^{(s)}(s))_u) \qquad (54)$$

This is one form of the BLT equation, with unknowns taken as the combined voltage waves leaving the junctions. Similarly, the BLT equation can be obtained with unknowns taken as the combined voltage waves entering the junctions. Rearranging Equation 51 gives

$$((\tilde{V}_n(0,s))_u) = ((\tilde{\Gamma}_{n,m}(s))_{u,v})^{-1} : ((\tilde{V}_n(L_u,s))_u) - ((\tilde{\Gamma}_{n,m}(s))_{u,v})^{-1} : ((\tilde{V}_n^{(s)}(s))_u)$$

$$(55)$$

Combining Equations 50 and 54 gives

$$\left[((\tilde{\Gamma}_{n,m}(s))_{u,v})^{-1} - ((\tilde{S}_{n,m}(s))_{u,v})\right] : ((V_n(L_u,s))_u)$$

$$= ((\tilde{\Gamma}_{n,m}(s))_{u,v})^{-1} : ((\tilde{V}_n^{(s)}(s))_u) \qquad (56)$$

Rearranging Equation 56 gives

$$((\tilde{V}_n(L_u,s))_u) = \left[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})\right]^{-1} : ((\tilde{V}_n^{(s)}(s))_u)$$

$$(57)$$

This is another form of the BLT equation with the unknown taken as the combined voltage waves entering the junctions. From Equations 54 and 57, the BLT equation can be derived in terms of the total voltage and total current supervectors. Note the order of multiplication of scattering and propagation supermatrices in Equations 54 and 57. Rearrange Equation 54 so that the order of multiplication

23

of matrices is the same as that in Equation 57. Equation 54 can be rearranged to give

$$((\tilde{V}_n(0,s))_u) = ((\tilde{S}_{n,m}(s))_{u,v}) : \left[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})\right]^{-1}$$
$$: ((\tilde{V}_n^{(s)}(s))_u) \tag{58}$$

From Equation 13 and 14, the supervectors for voltages and currents at the junction can be written in terms of the combined voltage waves leaving and entering junctions as

$$((\tilde{V}_n^{(0)}(s))_u) = \frac{1}{2}\left[((\tilde{V}_n(0,s))_u) + ((P_{n,m})_{u,v}) : ((\tilde{V}_n(L_u,s))_u)\right] \tag{59}$$

$$((\tilde{I}_n^{(0)}(s))_u) = \frac{1}{2}\ ((\tilde{Y}_{C_{n,m}}(s))_{u,v}) : \left[((\tilde{V}_n(0,s))_u) - ((P_{n,m})_{u,v}) : ((\tilde{V}_n(L_u,s))_u)\right] \tag{60}$$

where $\tilde{V}_n^{(0)}(s)$ and $\tilde{I}_n^{(0)}(s)$ are voltage and current on the nth conductor in the tube containing the uth wave at the junction from which the uth wave leaves.

In Equations 59 and 60, a permutation supermatrix $((P_{n,m})_{u,v})$ has been introduced to sum the appropriate outgoing and incoming waves at the junctions. The permutation supermatrix blocks have the following properties:

$$(P_{n,m})_{u,v} = \begin{cases} (1_{n,m})_{u,v} & \text{if } W_u \text{ and } W_v \text{ are on the same tube and } u \neq v \text{ (noting that this is a square matrix)} \\ (0_{n,m})_{u,v} & \text{if } W_u \text{ and } W_v \text{ are not on the same tube or } u = v \end{cases}$$

$$u,v = 1,2,\ldots,N_W$$

and

$$n = 1,2,\ldots,N_u$$
$$m = 1,2,\ldots,N_v \tag{61}$$

24

Only one block matrix $(P_{n,m})_{u,v}$ is equal to $(1_{n,m})$ on any row or column with respect to indices u or v. Thus $((P_{n,m})_{u,v})$ is an orthogonal *supermatrix*.

Substituting Equations 57 and 58 into Equations 59 and 60 we obtain

$$((\tilde{V}_n^{(0)}(s))_u) = \frac{1}{2}\left[((\tilde{S}_{n,m}(s))_{u,v}) + ((P_{n,m})_{u,v})\right]$$
$$: \left[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})\right]^{-1} : ((\tilde{V}_n^{(s)}(s))_u)$$

(62)

$$((\tilde{I}_n^{(0)}(s))_u) = \frac{1}{2}((\tilde{Y}_{c_{n,m}}(s))_{u,v}) : \left[((\tilde{S}_{n,m}(s))_{u,v}) - ((P_{n,m})_{u,v})\right]$$
$$: \left[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})\right]^{-1} : ((\tilde{V}_n^{(s)}(s))_u)$$

(63)

Equations 61 and 62 are two forms of the BLT equation in terms of the voltage and current supervectors at the junctions.

III. BOUNDS FOR SIGNALS ON A MULTICONDUCTOR CABLE NETWORK

Having derived the BLT equations for general multiconductor-line networks, one can now establish upper and lower bounds on combined voltages, voltages, and currents, using the norm concept discussed in Appendix A. The BLT equations give voltages, currents, and combined voltages at the junctions. From these, voltages and currents can be found essentially everywhere, including at the junction terminals and at arbitrary positions on the tubes. However, only the junctions will be used to establish bounds.

1. BOUNDS ON COMBINED VOLTAGES, VOLTAGES, AND CURRENTS

Taking the norm of both sides of Equation 58 gives

$$\| ((\tilde{V}_n(0,s))_u) \| = \| ((\tilde{S}_{n,m}(s))_{u,v}) : [((1_{n,m})_{u,v})$$

$$- ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]^{-1} : ((\tilde{V}_n^{(s)}(s))_u) \| \qquad (64)$$

Using Equation A6 in Equation 64 gives

$$\| ((\tilde{V}_n(0,s))_u) \| \leq \| (\tilde{S}_{n,m}(s))_{u,v}) \| \; \| [((1_{n,m})_{u,v})$$

$$- ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v}]^{-1} \| : \| ((\tilde{V}_n^{(s)}(s))_u \| \qquad (65)$$

Rearranging Equation 58 gives

$$[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})] : ((\tilde{S}_{n,m}(s))_{u,v})^{-1}$$

$$: ((\tilde{V}_n(0,s))_u) = ((\tilde{V}_n^{(s)}(s))_u) \qquad (66)$$

Taking the norm of both sides of Equation 66 and using Equation A6, one obtains

$$\|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})] : ((S_{n,m}(s))_{u,v})^{-1}\|$$

$$\cdot \|((\tilde{V}_n(0,s))_u)\| \geq \|((\tilde{v}_n^{(s)}(s))_u)\|$$

or

$$\|((\tilde{V}_n(0,s))_u)\|$$

$$\geq \frac{\|((\tilde{v}_n^{(s)}(s))_u)\|}{\|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})] : ((\tilde{S}_{n,m}(s))_{u,v})^{-1}\|}$$

(67)

Equations 65 and 67 give upper and lower bounds on the norm of the combined voltage supervector for all waves leaving junctions, in terms of the norms of other quantities, such as combined voltage source waves, scattering supermatrix, and propagation supermatrix.

Similarly, one can obtain upper and lower bounds on the norm of combined voltage supervector for waves entering junctions. Taking the norm of both sides of Equation 57 and using Equation A6, one gets

$$\|((\tilde{V}_n(L_u))_u)\| \leq \|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]^{-1}\|$$

$$\cdot \|((\tilde{v}_n^{(s)}(s))_u)\|$$

(68)

and

$$\|((\tilde{V}_n(L_u,s))_u)\| \geq \frac{\|((\tilde{v}_n^{(s)}(s))_u)\|}{\|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((S_{n,m}(s))_{u,v})]\|}$$

(69)

Similarly, from Equations 62 and 63, the upper and lower bounds on the norms of voltage and current supervectors are given by

$$\|((\tilde{v}_n^{(0)}(s))_u)\| \leq \frac{1}{2}\|[((\tilde{S}_{n,m}(s))_{u,v}) + ((P_{n,m})_{u,v})]\|$$

$$\cdot \|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]^{-1}\| \|((\tilde{v}_n^{(s)}(s))_u)\| \tag{70}$$

$$\|((\tilde{I}_n^{(0)}(s))_u)\| \leq \frac{1}{2}\|((\tilde{Y}_{c_{n,m}}(s))_{u,v})\| \|[((\tilde{S}_{n,m}(s))_{u,v}) - ((P_{n,m})_{u,v})]\|$$

$$\cdot \|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]^{-1}\| \|((\tilde{v}_n^{(s)}(s))_u)\| \tag{71}$$

$$\|((\tilde{v}_n^{(0)}(s))_u)\| \geq \frac{1}{2}\|((\tilde{v}_n(s))_u)\|/\left[\|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}):((\tilde{S}_{n,m}(s))_{u,v})]\right.$$

$$\left. : [((\tilde{S}_{n,m}(s))_{u,v}) + ((P_{n,m})_{u,v})]^{-1}\|\right] \tag{72}$$

$$\|((\tilde{I}_n^{(0)}(s))_u)\| \geq \frac{1}{2}\|((\tilde{v}_n^{(s)}(s))_u)\|/\left[\|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}):((\tilde{S}_{n,m}(s))_{u,v})]\right.$$

$$\left. : [((\tilde{S}_{n,m}(s))_{u,v}) - ((P_{n,m}))_{u,v})]^{-1} : ((\tilde{Z}_{c_{n,m}}(s))_{u,v})\|\right] \tag{73}$$

Equations 70 and 71 give upper bounds on the voltages and currents, respec-
tively, and Equations 72 and 73 give lower bounds on voltages and currents,
respectively.

Before evaluating these upper and lower bounds, we shall illustrate
what these bounds mean. The upper and lower bounds defined in Equations 65,
67, and 68 through 73 are upper and lower bounds on the norm of vectors.
In Appendix A, 1, 2, and ∞ norms for vectors and matrices are defined. The
above equations are valid for any norm as long as they are consistent on both

sides of the equations. For the purpose of bounding signal levels, ∞ norm for vectors is most appropriate, for it gives the magnitude of the largest element of a vector. For a voltage or current vector at a junction or at any point along the line, the ∞ norm gives the magnitude of the maximum conductor voltage or current (pin voltage or current at terminations). Thus an upper and lower bound on the ∞ norm of a vector gives, respectively, an upper and lower bound on the magnitude of the largest element of the vector. The lower bound should not be confused with the magnitude of the smallest element of the vector.

Since the 2 norm of a matrix is obtained from the knowledge of its eigenvalues, it can be evaluated from the characteristic properties of the matrix and will therefore be used for matrices in the evaluation of upper and lower bounds. Using ∞ and 2 norms and Equations A64 and A94 through A99, upper and lower bounds can be written for combined voltages, voltages, and currents as follows.

Using 2 norms on both sides of Equation 65 and substituting Equation A63, one gets

$$\| ((\tilde{V}_n(0,s))_u) \|_\infty \leq \| ((\tilde{S}_{n,m}(s))_{u,v}) \|_2$$

$$\cdot \| [((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]^{-1} \|_2 \, \| ((\tilde{v}_n^{(s)}(s))_u) \|_2$$

$$(74)$$

Substituting Equation A65 into Equation 74 gives

$$\| ((\tilde{V}_n(0,s))_u) \|_\infty \leq \sqrt{N_s} \, \| ((\tilde{S}_{n,m}(s))_{u,v}) \|_2$$

$$\cdot \| [((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]^{-1} \|_2 \, \| ((\tilde{v}_n^{(s)}(s))_u) \|_\infty$$

$$(75)$$

Where N_s is the dimension (numbers of components) of the source supervector.

29

A lower bound on the combined voltages for all waves leaving junctions is obtained by using ∞ norms on both sides of Equation 67 as

$$\|((\tilde{V}_n(0,s))_u)\|_\infty \geq \|((\tilde{V}_n^{(s)}(s))_u)\|_\infty \Big/ \left[\|[((1_{n,m})_{u,v})-((\tilde{\Gamma}_{n,m}(s))_{u,v}):((\tilde{S}_{n,m}(s))_{u,v})]\right.$$

$$\left. :((S_{n,m}(s))_{u,v})^{-1}\|_\infty\right] \tag{76}$$

Substituting Equations A6 and A98 into Equation 76 gives

$$\|((\tilde{V}_n(0,s))_u)\|_\infty \geq \|((\tilde{V}_n^{(s)}(s))_u)\|_\infty \Big/ \left[\sqrt{N_s}\|[((1_{n,m})_{u,v})-((\tilde{\Gamma}_{n,m}(s))_{u,v}):((\tilde{S}_{n,m}(s))_{u,v})]\|_2\right.$$

$$\left. \cdot \|((\tilde{S}_{n,m}(s))_{u,v})^{-1}\|_2\right] \tag{77}$$

where N_s is the size of the supermatrices in the denominator.

An upper bound on the combined voltages for all waves entering junctions is obtained from Equation 68 using Equation A63 as

$$\|((\tilde{V}_n(L_u,s))_u)\|_\infty \leq \|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]^{-1}\|_2$$

$$\cdot \|((\tilde{V}_n^{(s)}(s))_u)\|_2 \tag{78}$$

Substituting Equation A65 into Equation 78 gives

$$\|((\tilde{V}_n(L_u,s))_u)\|_\infty \leq \sqrt{N_s} \|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]^{-1}\|_2$$

$$\cdot \|((\tilde{V}_n^{(s)}(s))_u)\|_\infty \tag{79}$$

Similarly, a lower bound on the combined voltages waves for all waves entering junctions obtained from Equation 69 using Equation A98 as

$$\|((\tilde{V}_n(L_u,s))_u)\|_\infty \geq \frac{\|((\tilde{V}_n^{(s)}(s))_u)\|_\infty}{\|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]\|_\infty}$$

$$\geq \frac{\|((\tilde{V}_n^{(s)}(s))_u)\|_\infty}{\sqrt{N_s}\,\|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]\|_2}$$

$$\tag{80}$$

Similarly, an upper and lower bound on the voltages and currents ·at the junctions is obtained from Equations 70 through 73 using Equations A6, A63, A65 and A98 as

$$\|((\tilde{V}_n^{(0)}(s))_u)\|_\infty \leq \frac{1}{2}\|[((\tilde{S}_{n,m}(s))_{u,v}) + ((P_{n,m})_{u,v})]\|_2$$

$$\cdot\|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]^{-1}\|_2\|((\tilde{V}_n^{(s)}(s))_u)\|_2$$

$$\leq \frac{1}{2}\sqrt{N_s}\,\|[((\tilde{S}_{n,m}(s))_{u,v}) + ((P_{n,m})_{u,v})]\|_2$$

$$\cdot\|[((1_{n,m}_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]^{-1}\|_2\|((\tilde{V}_n^{(s)}(s))_u)\|_\infty$$

$$\tag{81}$$

$$\|((\tilde{V}_n^{(0)}(s))_u)\|_\infty \geq \frac{1}{2}\|((\tilde{V}_n^{(s)}(s))_u)\|_\infty \Big/ \Big[\|[((1_{n,m}_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]$$

$$: [((\tilde{S}_{n,m}(s))_{u,v}) + ((P_{n,m})_{u,v})]^{-1}\|_\infty\Big]$$

$$\geq \frac{1}{2}\|((\tilde{V}_n^{(s)}(s))_u)\|_\infty \Big/ \Big[\sqrt{N_s}\|[((1_{n,m}_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]\|_2$$

$$\cdot\|[((\tilde{S}_{n,m}(s))_{u,v}) + ((P_{n,m})_{u,v})]^{-1}\|_2\Big]$$

$$\tag{82}$$

$$\|((\tilde{I}_n^{(0)}(s))_u)\|_\infty \leq \frac{1}{2} \|((\tilde{Y}_{c_{n,m}}(s))_{u,v})\|_2 \; \|[((\tilde{S}_{n,m}(s))_{u,v}) - ((P_{n,m})_{u,v})]\|_2$$

$$\cdot \|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]^{-1}\|_2 \|((\tilde{v}_n^{(s)}(s))_u)\|_2$$

$$\leq \frac{1}{2} \sqrt{N_s} \; \|((\tilde{Y}_{c_{n,m}}(s))_{u,v})\|_2 \|[((\tilde{S}_{n,m}(s))_{u,v}) - ((P_{n,m})_{u,v})]\|_2$$

$$\cdot \|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]^{-1}\|_2 \|((\tilde{v}_n^{(s)}(s))_u)\|$$

$$(83)$$

$$\|((\tilde{I}_n^{(0)}(s))_u)\|_\infty \geq \frac{1}{2} \|((\tilde{v}_n^{(s)}(s))_u)\|_\infty / \left[\| [((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})] \right.$$

$$\cdot [((\tilde{S}_{n,m}(s))_{u,v}) - ((P_{n,m})_{u,v})]^{-1} : ((\tilde{Z}_{c_{n,m}}(s))_{u,v})\|_\infty \Big]$$

$$\geq \frac{1}{2} \|((\tilde{v}_n^{(s)}(s))_u)\|_\infty /$$

$$/ \sqrt{N_s} \; \|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]\|_2$$

$$\cdot \|[((\tilde{S}_{n,m}(s))_{u,v}) - ((P_{n,m})_{u,v})]^{-1}\|_2 \|((\tilde{Z}_{c_{n,m}}(s))_{u,v})\|_2 \Big]$$

$$(84)$$

So far, relations for upper and lower bounds for combined voltages, voltages, and currents have been derived in terms of norms of other parameters, such as the propagation and scattering supermatrices, the characteristic impedance or admittance supermatrix, and the source supervector. Thus, to establish upper and lower bounds on combined voltages, voltages, and currents, one has to first establish bounds on the parameters. To establish bounds on the parameters and the sources for a general multiconductor cable network is very difficult. Furthermore, if such bounds could be established on parameters, the resulting bounds on the voltages and currents may be unrealistic. To get reasonable bounds, special canonical configurations of a multiconductor cable network are considered in Sections IV and V.

32

The simplest of these configurations is a uniform section of a multi-conductor transmission line in a homogeneous medium terminated at both ends and excited by an external field or voltages and currents at terminations. To make the transmission line configuration more complex, a branch can be added to a uniform section of the line. The branched line will illustrate the procedure for calculation of bounds for cable networks with junctions. These two configurations will be considered in Sections IV and V.

2. BOUNDS IN TERMS OF BULK CURRENT

In evaluating the EMP vulnerability of a system, the bounds which are of most interest are the bounds on pin currents in terms of the bulk current. The bulk current on a multiconductor transmission line is defined as the algebraic sum of all the wire currents at a given cross section. This concept of pin current bounding in terms of the bulk current has tremendous implications for aicraft testing. If such a bound can be established, then one need measure only bulk currents on cables in an aircraft, thereby reducing the number of measurements by orders of magnitude. This section addresses the problem.

Since the 1 norm of a vector is defined as the sum of the magnitudes of its components, and the bulk current is the algebraic sum of the wire current in a cable, then for current vector on a multiconductor line at a termination one has

$$\|(\tilde{I}_n^{(0)}(s))_{r;\nu}\|_1 \geq |\tilde{I}_B^{(0)}(s)_{r;\nu}| \tag{85}$$

where $(\tilde{I}_n^{(0)}(s))_{r;\nu}$ is the current vector for the rth tube at the νth junction, and $\tilde{I}_B^{(0)}(s)_{r;\nu}$ is the bulk current on the rth tube at the νth junction and is defined as $\tilde{I}_B^{(0)}(s)_{r;\nu} \equiv \sum_{n=1}^{N_u} \tilde{I}_n^{(0)}(s)_{r;\nu}$.

33

Writing Equation 85 for currents at all the junctions gives

$$\|((\tilde{I}_n^{(0)}(s))_u)\|_1 \geq \left| \sum_{r=1}^{N_u} \sum_{\nu=1}^{\nu} \tilde{I}_B^{(0)}(s)_{r;\nu} \right| \equiv |\tilde{I}_B| \tag{86}$$

where the right-hand side is the sum of all the bulk currents in all the tubes at all the junctions.

Now express the ∞ norm of the current supervector in terms of the total bulk current. This can be further decomposed in terms of bulk currents on tubes at various junctions for specific problems. Substituting Equation A63 into Equation 84 gives

$$\|((\tilde{I}_n^{(0)}(s))_u)\|_1 \geq \frac{1}{2} \|((\tilde{v}_n^{(s)}(s))_u)\|_\infty$$

$$\Big/ \left[\sqrt{N_s} \| [((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})] \|_2 \right.$$

$$\left. \cdot \| [((\tilde{S}_{n,m}(s))_{u,v}) - ((P_{n,m})_{u,v})]^{-1} \|_2 \|((\tilde{Z}_{c_{n,m}}(s))_{u,v})\|_2 \right] \tag{87}$$

and taking 1 norm of both sides of Equation 63, and then substituting Equations A6 and A94 into the result gives

$$\|((\tilde{I}_n^{(0)}(s))_u)\|_1 \leq \frac{1}{2} \sqrt{N_s} \|((\tilde{Y}_{c_{n,m}}(s))_{u,v})\|_2 \| [((\tilde{S}_{n,m}(s))_{u,v}) - ((P_{n,m})_{u,v})] \|_2$$

$$\cdot \| [((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m})_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]^{-1} \|_2 \|((\tilde{v}_n^{(s)}(s))_u)\|_1 \tag{88}$$

In Equations 87 and 88, 2 norms of matrices are used, since these can be computed from energy conservation. Dividing Equation 84 by Equation 88 and then substituting Equation 86 in the result, one obtains

34

$$\frac{\|((\tilde{I}_n^{(0)}(s))_u)\|_\infty}{|\tilde{I}_B|}$$

$$\geq \|((\tilde{V}_n^{(s)}(s))_u\|_\infty / [N_s \| [((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v} : ((\tilde{S}_{n,m}(s))_{u,v})]\|_2$$

$$\cdot \| [((\tilde{S}_{n,m}(s))_{u,v}) - ((P_{n,m})_{u,v})]^{-1}\|_2 \ \|((\tilde{Z}_{c_{n,m}}(s))_{u,v})\|_2$$

$$\cdot \ \|((\tilde{Y}_{c_{n,m}}(s))_{u,v})\|_2 \ \| [((\tilde{S}_{n,m}(s))_{u,v}) - ((P_{n,m})_{u,v})]\|_2$$

$$\cdot \ \| [((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v} : ((\tilde{S}_{n,m}(s))_{u,v})]^{-1}\|_2 \ \|((\tilde{V}_n^{(s)}(s))_u)\|_1]$$

$$\tag{89}$$

Thus Equation 89 gives a lower bound on the ratio of the maximum pin current to the bulk current.

Similarly, dividing Equation 83 by Equation 87 gives

$$\frac{\|((\tilde{I}_n^{(0)}(s))_u)\|_\infty}{\|((\tilde{I}_n^{(0)}(s))_u)\|_1}$$

$$\leq \ N_s \ \|((\tilde{Y}_{c_{n,m}}(s))_{u,v})\|_2 \| [((\tilde{S}_{n,m}(s))_{u,v}) - ((P_{n,m})_{u,v})]\|_2$$

$$\cdot \ \| [((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v} : ((\tilde{S}_{n,m}(s))_{u,v})]^{-1}\|_2 \|((\tilde{Z}_{c_{n,m}}(s))_{u,v})\|_2$$

$$\cdot \ \| [((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v} : ((\tilde{S}_{n,m}(s))_{u,v})]\|_2 \| [((\tilde{S}_{n,m}(s))_{u,v}) - ((P_{n,m})_{u,v})]^{-1}\|_2$$

$$\tag{90a}$$

Also, from Equation A64

$$\frac{\|((\tilde{I}_n^{(0)}(s))_u)\|_\infty}{\|((\tilde{I}_n^{(0)}(s))_u)\|_1} \leq 1 \tag{90b}$$

35

In Equation 90 the upper bound is independent of sources. In Equations 89 and 90, the norms of matrices and their inverses occur in pairs. From Equation A56, the product of the norm of a matrix and the norm of its inverse is greater than or equal to 1.

From physical principles and Equation A65 the lower bound on the ratio of the maximum pin current to the bulk current is $1/N_s$. Thus, with the result in Equation A56 in mind, the lower bound in Equation 89 is not useful since it gives a lower bound less than $1/N_s$.

Equation 90 gives an upper bound on the ratio of the maximum pin current to the sum of the magnitudes of all the pin currents. Since we cannot substitute the 1 norm in the denominator with the bulk current, this bound is not very useful either. It is obvious from the above discussion that an upper bound on the ratio of the maximum pin current to the bulk current cannot be obtained analytically. However, it is seen easily that, in general, pin current is not bounded with respect to bulk current, since the bulk current in a cable can be zero, while the individual pin currents are nonzero; for example, a two-wire cable excited in the differential mode has nonzero pin current and zero bulk current.

IV. BOUNDS FOR A UNIFORM SECTION OF A MULTICONDUCTOR TRANSMISSION LINE

This section covers a special case of a general multiconductor cable network, a uniform section of a multiconductor transmission line terminated at both ends. Two types of excitations will be considered. In the first type of excitation, the line is excited by an incident external field, and in the second type, the line is excited by voltage or current sources at the terminations.

Consider a multiconductor transmission line formed by N conductors plus a reference conductor or ground as shown in Figure 3. The line is assumed to be uniform along its length (z coordinate), but with arbitrary cross section. In general, the dielectric surrounding the line is inhomogeneous (e.g., cable made of insulated conductors having different geometries and dielectric materials).

The wave traveling in +z direction is denoted by wave W_1 or simply wave 1, and the wave traveling in -z direction as W_2 or wave 2, as shown in Figure 3. Then the combined voltage vectors for multiconductor transmission-line in Figure 3 are given by

$$((\tilde{V}_n(0,s))_u) = \begin{pmatrix} (\tilde{V}_n(0,s))_1 \\ (\tilde{V}_n(0,s))_2 \end{pmatrix} \tag{91}$$

$$((\tilde{V}_n(L_u,s))_u) = \begin{pmatrix} (\tilde{V}_n(L,s))_1 \\ (\tilde{V}_n(L,s))_2 \end{pmatrix} \tag{92}$$

where $(\tilde{V}_n(0,s))_1$ and $(\tilde{V}_n(0,s))_2$ are the waves leaving junctions at z = 0 and z = L, respectively, and $(\tilde{V}_n(L,s))_1$ and $(\tilde{V}_n(L,s))_2$ are the waves entering junctions at z = L and z = 0, respectively.

37
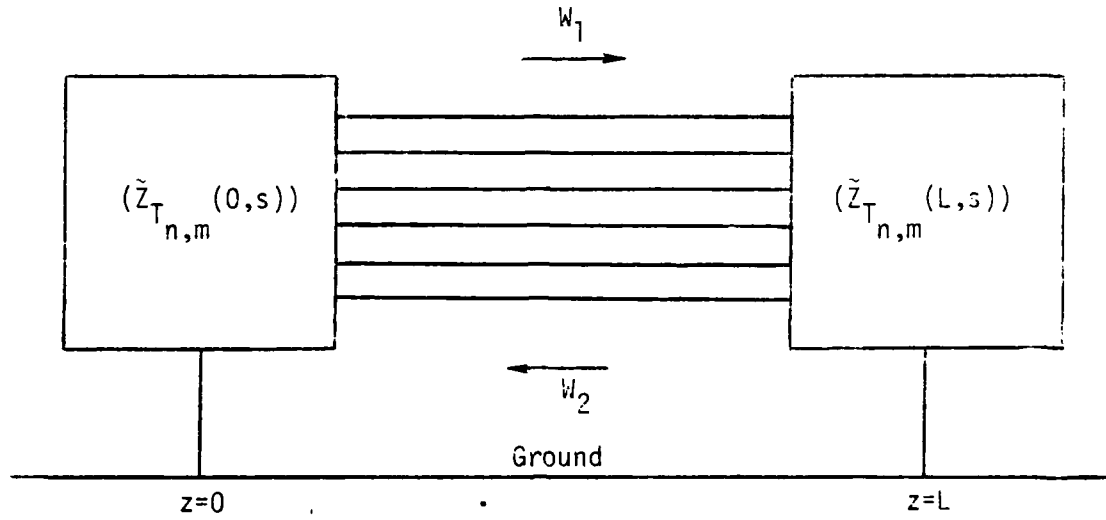
Figure 3. A multiconductor transmission line over a ground plane,
terminated at both ends.

The waves leaving and entering junctions are related through scattering
matrices as

$$(\tilde{V}_n(0,s))_1 = (\tilde{S}_{n,m}(s))_{1,2} \cdot (\tilde{V}_n(L,s))_2 \tag{93}$$

$$(\tilde{V}_n(0,s))_2 = (\tilde{S}_{n,m}(s))_{2,1} \cdot (\tilde{V}_n(L,s))_1 \tag{94}$$

where $(\tilde{S}_{n,m}(s))_{1,2}$ and $(\tilde{S}_{n,m}(s))_{2,1}$ are scattering matrices of junctions at
$z = 0$ and $z = L$, respectively. The subscripts 1,2 and 2,1 indicate that the
2 waves is scattered into the 1 wave and the 1 wave is scattered into the 2
wave, respectively. Combining Equations 93 and 94 and writing the scattering
matrices in supermatrix form gives

$$\begin{pmatrix} (\tilde{V}_n(0,s))_1 \\ (\tilde{V}_n(0,s))_2 \end{pmatrix} = \begin{pmatrix} (0_{n,m})_{1,1} & (\tilde{S}_{n,m}(s))_{1,2} \\ (\tilde{S}_{n,m}(s))_{2,1} & (0_{n,m})_{2,2} \end{pmatrix} : \begin{pmatrix} (\tilde{V}_n(L,s))_1 \\ (\tilde{V}_n(L,s))_2 \end{pmatrix} \tag{95}$$

or

$$((\tilde{V}_n(0,s))_u) = ((\tilde{S}_{n,m}(s))_{u,v}) : ((\tilde{V}_n(L,s))_u) \qquad (96)$$

where

$$((\tilde{S}_{n,m}(s))_{u,v}) = \begin{pmatrix} (0_{n,m}) & (\tilde{S}_{n,m}(s))_{1,2} \\ (\tilde{S}_{n,m}(s))_{2,1} & (0_{n,m}) \end{pmatrix} \qquad (97)$$

$$\equiv \text{ scattering supermatrix}$$

$$u = 1,2$$

$$v = 1,2$$

$$n = m = 1,2,\ldots,N$$

$(\tilde{S}_{n,m}(s))_{1,2}$ and $(\tilde{S}_{n,m}(s))_{2,1}$ are the reflection coefficient matrices at $z = 0$ and $z = L$, respectively, and are given by Equations 23 through 26.

From Equation 43, we can write the relation between the 1 wave at $z = L$ in terms of the 1 wave at $z = 0$ as

$$(\tilde{V}_n(L,s))_1 = \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))L\} \cdot (\tilde{V}_n(0,s))_1$$

$$+ \int_0^L \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))[L - z']\} \cdot (\tilde{v}_n^{(s)'}(z',s))_1 dz' \qquad (98)$$

Note that

$$z_u = \begin{cases} z & \text{for } u = 1 \\ L-z & \text{for } u = 2 \end{cases}$$

Similarly, the 2 wave at $z = L$ can be expressed in terms of the 2 wave at $z = 0$

$$(\tilde{V}_n(L,s))_2 = \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))L\} \cdot (\tilde{V}_n(0,s))_2$$

$$- \int_0^L \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))z''\} \cdot (\tilde{V}_n^{(s)'}(z'',s))_2 dz'' \qquad (99)$$

where $z'' = L - z'$. Combining Equations 98 and 99 gives

$$\begin{pmatrix} (\tilde{V}_n(L,s))_1 \\ (\tilde{V}_n(L,s))_2 \end{pmatrix} = \begin{pmatrix} \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))L\} & (0_{n,m}) \\ (0_{n,m}) & \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))L\} \end{pmatrix} : \begin{pmatrix} (\tilde{V}_n(0,s))_1 \\ (\tilde{V}_n(0,s))_2 \end{pmatrix}$$

$$+ \begin{pmatrix} \int_0^L \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))[L - z']\} \cdot (\tilde{V}_n^{(s)'}(z',s))_1 dz' \\ -\int_0^L \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))z''\} \cdot (\tilde{V}_n^{(s)'}(z'',s))_2 dz'' \end{pmatrix} \qquad (100)$$

Then Equation 100 can be written in supermatrix notation as

$$((\tilde{V}_n(L,s))_u) = ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{V}_n(0,s))_u) + ((\tilde{V}_n^{(s)}(s))_u) \qquad (101)$$

where

$$((\tilde{\Gamma}_{n,m}(s))_{u,v}) = \begin{pmatrix} (\tilde{\Gamma}_{n,m}(s))_{1,1} & (0_{n,m}) \\ (0_{n,m}) & (\tilde{\Gamma}_{n,m}(s))_{2,2} \end{pmatrix} \qquad (102)$$

$$(\tilde{\Gamma}_{n,m}(s))_{1,1} = (\tilde{\Gamma}_{n,m}(s))_{2,2} = (\tilde{\Gamma}_{n,m}(s)) = \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))L\} \qquad (103)$$

40

$$((\tilde{V}_n^{(s)}(s))_u) = \begin{pmatrix} \int_0^L \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))[L - z']\} \cdot (\tilde{V}_n^{(s)'}(z',s))_1 dz' \\ -\int_0^L \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))z''\} \cdot (\tilde{V}_n^{(s)'}(z',s))_2 dz'' \end{pmatrix} \quad (104)$$

Substituting Equations 11 and 12 into Equation 104, one gets the source supervector for the two waves in terms of the voltage and current source vectors as

$$((\tilde{V}_n^{(s)}(s))_u)$$

$$= \begin{pmatrix} \int_0^L \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))[L-z']\} \cdot [(\tilde{V}_n^{(s)'}(z',s)) + (\tilde{Z}_{c_{n,m}}(s)) \cdot (\tilde{I}_n^{(s)'}(z',s))]dz' \\ -\int_0^L \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))z''\} \cdot [(\tilde{V}_n^{(s)'}(z'',s)) - (\tilde{Z}_{c_{n,m}}(s)) \cdot (\tilde{I}_n^{(s)'}(z'',s))]dz'' \end{pmatrix}$$

$$(105)$$

where $(\tilde{I}_n^{(s)'}(z',s))$ is now taken positive in W_1 (or $+z$) direction.

1. NORM OF THE SCATTERING SUPERMATRIX

The scattering supermatrix for a uniform section of a multiconductor transmission line is given by Equation 97, and has its diagonal block matrices as null matrices and off-diagonal block matrices as the reflection coefficient matrices at $z = 0$ and $z = L$. From Equation 97, one can write

41

$$((\tilde{S}_{n,m}(s))_{u,v})^\dagger : ((\tilde{S}_{n,m}(s))_{u,v})$$

$$= \begin{pmatrix} (\tilde{S}_{n,m}(s))_{2,1}^\dagger \cdot (\tilde{S}_{n,m}(s))_{2,1} & (0_{n,m}) \\ (0_{n,m}) & (\tilde{S}_{n,m}(s))_{1,2}^\dagger \cdot (\tilde{S}_{n,m}(s))_{1,2} \end{pmatrix}$$

$$\equiv \left[(\tilde{S}_{n,m}(s))_{2,1}^\dagger \cdot (\tilde{S}_{n,m}(s))_{2,1}\right] \oplus \left[(\tilde{S}_{n,m}(s))_{1,2}^\dagger \cdot (\tilde{S}_{n,m}(s))_{1,2}\right] \tag{106}$$

The supermatrix in Equation 106 is block diagonal and, therefore, its eigenvalues are the eigenvalues of its block matrices. Equations 106 and A73 give

$$\|((\tilde{S}_{n,m}(s))_{u,v})\|_2 = \left[\lambda_{max} \left\{ \begin{array}{c} (\tilde{S}_{n,m}(s))_{2,1}^\dagger \cdot (\tilde{S}_{n,m}(s))_{2,1} \\ (\tilde{S}_{n,m}(s))_{1,2}^\dagger \cdot (\tilde{S}_{n,m}(s))_{1,2} \end{array} \right\} \right]^{1/2}$$

$$= \max_{\substack{u=1,v=2 \\ u=2,v=1}} \|(\tilde{S}_{n,m}(s))_{u,v}\|_2 \tag{107}$$

Thus the 2 norm of the scattering supermatrix of a uniform section of a multiconductor line is the larger of the 2 norms of the scattering matrices at the terminations.

For passive terminations, an upper bound can be established on the 2 norm of the scattering supermatrix. For passive terminations, the 2 norms of the scattering matrices (reflection coefficient matrices) $(\tilde{S}_{n,m}(s))_{1,2}$ and $(\tilde{S}_{n,m}(s))_{2,1}$ satisfy the inequality

$$\|(\tilde{S}_{n,m}(s))\|_2 \leq 1 \qquad \text{for } s = j\omega \tag{108}$$

The proof of Equation 108 is illustrated in Appendix B.

Similarly, one can establish a lower bound on the norm of the inverse of the scattering supermatrix for passive terminations.

From Equation 97, one can write the inverse of the scattering supermatrix as

$$((\tilde{S}_{n,m}(s))_{u,v})^{-1} = \begin{pmatrix} (0_{n,m}) & (\tilde{S}_{n,m}(s))^{-1}_{1,2} \\ (\tilde{S}_{n,m}(s))^{-1}_{2,1} & (0_{n,m}) \end{pmatrix} \qquad (109)$$

Since $(\tilde{S}_{n,m}(s))_{1,2}$ and $(\tilde{S}_{n,m}(s))_{2,1}$ are square matrices for a uniform section of a line. Then from Equation 109

$$((\tilde{S}_{n,m}(s))_{u,v})^{-1\dagger} : ((\tilde{S}_{n,m}(s))_{u,v})^{-1}$$

$$= \begin{pmatrix} (\tilde{S}_{n,m}(s))^{-1\dagger}_{2,1} \cdot (\tilde{S}_{n,m}(s))^{-1}_{2,1} & (0_{n,m}) \\ (0_{n,m}) & (\tilde{S}_{n,m}(s))^{-1\dagger}_{1,2} \cdot (\tilde{S}_{n,m}(s))^{-1}_{1,2} \end{pmatrix}$$

$$\equiv \left[(\tilde{S}_{n,m}(s))^{-1\dagger}_{2,1} \cdot (\tilde{S}_{n,m}(s))^{-1}_{2,1}\right] \oplus \left[(\tilde{S}_{n,m}(s))^{-1\dagger}_{1,2} \cdot (\tilde{S}_{n,m}(s))^{-1}_{1,2}\right]$$

$$(110)$$

From Equations 110 and A73, one has

$$\|((\tilde{S}_{n,m}(s))_{u,v})^{-1}\|_2 = \left[\lambda_{max}\left\{ \begin{matrix} (\tilde{S}_{n,m}(s))^{-1\dagger}_{2,1} \cdot (\tilde{S}_{n,m}(s))^{-1}_{2,1} \\ (\tilde{S}_{n,m}(s))^{-1\dagger}_{1,2} \cdot (\tilde{S}_{n,m}(s))^{-1}_{1,2} \end{matrix}\right\}\right]^{1/2}$$

$$= \max_{\substack{u=1,v=2 \\ u=2,v=1}} \|(\tilde{S}_{n,m}(s))^{-1}_{u,v}\|_2 \qquad (111)$$

The 2 norm of the inverse of the scattering supermatrix is greater than or equal to one for $s = j\omega$ (see Appendix B).

43

2. NORM OF THE PROPAGATION SUPERMATRIX

The propagation supermatrix for a uniform section of a multiconductor transmission line in Equation 102 is block diagonal, with block matrices equal to the propagation matrix of the line given in Equation 103. From Equation A73, one can write the 2 norm of the scattering supermatrix as

$$\| ((\tilde{\Gamma}_{n,m}(s))_{u,v} \|_2 = \| (\tilde{\Gamma}_{n,m}(s)) \|_2$$

$$= \| \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))L\} \|_2 \qquad (112)$$

Thus the 2 norm of the propagation supermatrix is equal to the 2 norm of the propagation matrix $(\tilde{\Gamma}_{n,m}(s))$ of the line.

The propagation matrix $(\tilde{\Gamma}_{n,m}(s))$ is a complex, nonsymmetric matrix in general. The calculation of the eigenvalues of the propagation matrix requires knowledge of the propagation modes, eigenvalues, and eigenvectors of the characteristic propagation matrix $(\tilde{\gamma}_{c_{n,m}}(s))$. Since it is difficult to find eigenvectors of the propagation matrix without the complete knowledge of the matrix itself, for the purpose of establishing bounds the investigation will be limited to a homogeneous medium case. For a multiconductor transmission line surrounded by a homogeneous medium, the characteristic propagation matrix is diagonal with equal elements since all the modes propagate with the same speed. The diagonal elements of the characteristic propagation matrix for a homogeneous passive case are given by

$$\tilde{\gamma}_{c_{n,n}}(s) = \tilde{\alpha}(s) + j\tilde{\beta}(s) \qquad \text{for } s = j\omega \qquad (113)$$
$$\tilde{\alpha}(s) \geq 0$$

where α and β are the attenuation and phase constants.

44

From Equation 113, diagonal terms of the matrix $(\tilde{\Gamma}_{n,m}(s))$ are given by

$$\tilde{\Gamma}_{n,n}(s) = \exp\{-[\tilde{\alpha}(s) + j\tilde{\beta}(s)]L\}$$

$$= \exp\{-\tilde{\alpha}(s)L\} \exp\{-j\tilde{\beta}(s)L\} \qquad \text{for } s = j\omega \qquad (114)$$

From Equation 114, the 2 norm of the matrix $(\tilde{\Gamma}_{n,m}(s))$ can be written as

$$\|(\tilde{\Gamma}_{n,m}(s))\|_2 = \|(\tilde{\Gamma}_{n,m}(s))\| = |\exp\{-\alpha(s)L\} \exp\{-j\beta(s)L\}|$$

$$= \exp\{-\alpha(s)L\} \qquad \text{for } s = j\omega \qquad (115)$$

since the magnitude of the second exponential term is equal to one.

From Equation 115 one can conclude that

$$\|(\tilde{\Gamma}_{n,m}(s))\|_2 \leq 1 \qquad \text{for } s = j\omega \qquad (116)$$

and hence

$$\|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2 \leq 1 \qquad \text{for } s = j\omega \qquad (117)$$

3. NORM OF THE SOURCE SUPERVECTOR

The source supervector is given by Equation 105 and, using Equation A62, its norm can be expressed as

$$\|((\tilde{v}_n^{(s)}(s))_u)\|$$

$$= \left\| \left(\begin{array}{c} \|\int_0^L \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))[L - z']\} \cdot [(\tilde{v}_n^{(s)'}(z',s)) + (\tilde{Z}_{c_{n,m}}(s)) \cdot (\tilde{I}_n^{(s)'}(z',s))]dz'\| \\ \\ \| -\int_0^L \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))z''\} \cdot [(\tilde{v}_n^{(s)'}(z'',s)) - (\tilde{Z}_{c_{n,m}}(s)) \cdot (\tilde{I}_n^{(s)'}(z'',s))]dz''\| \end{array} \right) \right\|$$

$$(118)$$

Using Equations A91 and A6, the norm of the vectors in Equation 118 can be expressed as

$$\left\| \int_0^L \exp\{-(\tilde{\gamma}_{C_{n,m}}(s))[L-z']\} \cdot [\tilde{V}_n^{(s)'}(z',s) + (\tilde{Z}_{C_{n,m}}(s)) \cdot (\tilde{I}_n(z',s))]dz' \right\|$$

$$\leq \int_0^L \|\exp\{-(\tilde{\gamma}_{C_{n,m}}(s))[L-z']\}\| \, \|[(\tilde{V}_n^{(s)'}(z',s)) + (\tilde{Z}_{C_{n,m}}(s)) \cdot (\tilde{I}_n(z',s))]\| dz'$$

$$\leq \int_0^L \|\exp\{-(\tilde{\gamma}_{C_{n,m}}(s))[L-z']\}\| \; [\|\tilde{V}_n^{(s)'}(z',s)\| + \|(\tilde{Z}_{C_{n,m}}(s))\| \, \|(\tilde{I}_n(z',s))\|]dz'$$

$$(119)$$

$$\left\| -\int_0^L \exp\{-(\tilde{\gamma}_{C_{n,m}}(s))z''\} \cdot [(\tilde{V}_n^{(s)'}(z'',s)) - (\tilde{Z}_{C_{n,m}}(s)) \cdot (\tilde{I}_n^{(s)'}(z'',s))]dz'' \right\|$$

$$\leq \int_0^L \|\exp\{-(\tilde{\gamma}_{C_{n,m}}(s))z''\}\| \, \|[(\tilde{V}_n^{(s)'}(z'',s)) - (\tilde{Z}_{C_{n,m}}(s)) \cdot (\tilde{I}_n^{(s)'}(z'',s))]\| dz''$$

$$\leq \int_0^L \|\exp\{-(\tilde{\gamma}_{C_{n,m}}(s))z''\}\| \; [\|(\tilde{V}_n^{(s)'}(z'',s))\| + \|(\tilde{Z}_{C_{n,m}}(s))\| \, \|(\tilde{I}_n^{(s)'}z'',s))\|]dz''$$

$$(120)$$

For a homogeneous medium, from Equation 115 the norm of the propagation matrix is bounded by (for $s = j\omega$)

$$\|\exp\{-(\tilde{\gamma}_{C_{n,m}}(s))[L - z']\| \leq 1 \tag{121}$$

$$\|\exp\{-(\tilde{\gamma}_{C_{n,m}}(s))z''\| \leq 1 \tag{122}$$

Substituting Equations 119 through 122 into Equation 118 gives

$$\|((\tilde{V}_n^{(s)}(s))_u)\| \leq \left\| \begin{pmatrix} \int_0^L [\|(\tilde{V}_n^{(s)'}(z',s))\| + \|(\tilde{Z}_{C_{n,m}}(s))\| \, \|(\tilde{I}_n^{(s)'}(z',s))\|]dz' \\ \int_0^L [\|(\tilde{V}_n^{(s)'}(z'',s))\| + \|(\tilde{Z}_{C_{n,m}}(s))\| \, \|(\tilde{I}_n^{(s)'}(z'',s))\|]dz'' \end{pmatrix} \right\|$$

$$(123)$$

46

Note that the two integrals are equal so that only one is needed, and the norms can be expressed in terms of the norm of this one as

$$\|((\tilde{V}_n^{(s)}(s))_u)\|_1 \leq 2\left[\int_0^L [\|(\tilde{V}_n^{(s)'}(z',s))\|_1 + \|(\tilde{Z}_{c_{n,m}}(s))\|_1 \|(\tilde{I}_n^{(s)'}(z',s))\|_1]dz'\right]$$

(124a)

$$\|((\tilde{V}_n^{(s)}(s))_u)\|_2 \leq \sqrt{2}\left[\int_0^L [\|(\tilde{V}_n^{(s)'}(z',s))\|_2 + \|(\tilde{Z}_{c_{n,m}}(s))\|_2 \|(\tilde{I}_n^{(s)'}(z',s))\|_2]dz'\right]$$

(124b)

$$\|((\tilde{V}_n^{(s)}(s))_u)\|_\infty \leq \int_0^L [\|(\tilde{V}_n^{(s)'}(z',s))\|_\infty + \|(\tilde{Z}_{c_{n,m}}(s))\|_\infty \|(\tilde{I}_n^{(s)'}(z',s))\|_\infty]dz'$$

(124c).

If the per-unit-length voltage and current source vectors along the line can be expressed using delta functions as

$$(\tilde{V}_n^{(s)'}(z',s)) = \sum_{\sigma=1}^{\sigma_{max}} (\tilde{V}_n^{(s)}(s))_\sigma \delta(z'-\zeta_\sigma) \quad , \quad (\tilde{I}_n^{(s)'}(z',s)) = \sum_{\sigma=1}^{\sigma_{max}} (\tilde{I}_n^{(s)}(s))_\sigma \delta(z'-\zeta_\sigma)$$

where $\sigma = 1,2,\ldots,\sigma_{max}$, then Equations 124a through c can be written as

$$\|((\tilde{V}_n^{(s)}(s))_u)\|_1 \leq 2 \sum_{\sigma=1}^{\sigma_{max}} [\|(\tilde{V}_n^{(s)}(s))_\sigma\|_1 + \|(\tilde{Z}_{c_{n,m}}(s))\|_1 \|(\tilde{I}_n^{(s)}(s))_\sigma\|_1]$$

(124d)

$$\|((\tilde{V}_n^{(s)}(s))_u)\|_2 \leq \sqrt{2} \sum_{\sigma=1}^{\sigma_{max}} [\|(\tilde{V}_n^{(s)}(s))_\sigma\|_2 + \|(\tilde{Z}_{c_{n,m}}(s))\|_2 \|(\tilde{I}_n^{(s)}(s))_\sigma\|_2]$$

(124e)

$$\|((\tilde{V}_n^{(s)}(s))_u)\|_\infty \leq \sum_{\sigma=1}^{\sigma_{max}} [\|(\tilde{V}_n^{(s)}(s))_\sigma\|_\infty + \|(\tilde{Z}_{c_{n,m}}(s))\|_\infty \|(\tilde{I}_n^{(s)}(s))_\sigma\|_\infty]$$

(124f)

Equations 124a through 124f express the norms of the source supervector in terms of the norms of the per-unit-length voltage and current

source vectors on the line and the norms of the characteristic-impedance matrix of the line. The expressions for the norms of the source supervector can be simplified for the following three special cases.

a. Sources are delta functions; that is, the sources exist only at a point along the line (localized sources). In this case the 1, 2, and ∞ norms of the source supervector in Equation 124 reduce to

$$\| ((\tilde{V}_n^{(s)}(s))_u) \|_1 \leq 2 \| (\tilde{V}_n^{(s)}(s)) \|_1 + 2 \| (\tilde{Z}_{c_{n,m}}(s)) \|_1 \| (\tilde{I}_n^{(s)}(s)) \|_1 \qquad (125a)$$

$$\| ((\tilde{V}_n^{(s)}(s))_u) \|_2 \leq \sqrt{2} \| (\tilde{V}_n^{(s)}(s)) \|_2 + \sqrt{2} \| (\tilde{Z}_{c_{n,m}}(s)) \|_2 \| (\tilde{I}_n^{(s)}(s)) \|_2 \qquad (125b)$$

$$\| ((\tilde{V}_n^{(s)}(s))_u) \|_\infty \leq \| (\tilde{V}_n^{(s)}(s)) \|_\infty + \| (\tilde{Z}_{c_{n,m}}(s)) \|_\infty \| (\tilde{I}_n^{(s)}(s)) \|_\infty \qquad (125c)$$

b. Sources are uniform along the line. In this case the 1, 2, and ∞ norms of the source supervector in Equation 124 reduce to

$$\| ((\tilde{V}_n^{(s)}(s))_u) \|_1 \leq 2L \| (\tilde{V}_n^{(s)'}(s)) \|_1 + 2L \| (\tilde{Z}_{c_{n,m}}(s)) \|_1 \| (\tilde{I}_n^{(s)'}(s)) \|_1 \qquad (126a)$$

$$\| ((\tilde{V}_n^{(s)}(s))_u) \|_2 \leq \sqrt{2} L \| (\tilde{V}_n^{(s)'}(s)) \|_2 + \sqrt{2} L \| (\tilde{Z}_{c_{n,m}}(s)) \|_2 \| (\tilde{I}_n^{(s)'}(s)) \|_2$$
$$(126b)$$

$$\| ((\tilde{V}_n^{(s)}(s))_u) \|_\infty \leq L \| (\tilde{V}_n^{(s)'}(s)) \|_\infty + L \| (\tilde{Z}_{c_{n,m}}(s)) \|_\infty \| (\tilde{I}_n^{(s)'}(s)) \|_\infty \qquad (126c)$$

c. Sources are rather uniform; that is, the variation of per-unit-length sources along the line is small. In this case, it is appropriate to use the maximum so that the 1, 2, and ∞ norms of the source supervector can be written as

$$\|((\tilde{v}_n^{(s)}(s))_u)\|_1 \leq 2L[\ \|(\tilde{v}_n^{(s)'}(z,s))\|_1 + \|(\tilde{Z}_{c_{n,m}}(s))\|_1 \ \|(\tilde{I}_n^{(s)'}(s))\|_1]_{max}$$

<div align="right">(127a)</div>

$$\|((\tilde{v}_n^{(s)}(s))_u)\|_2 \leq \sqrt{2}\ L[\ \|(\tilde{v}_n^{(s)'}(z,s))\|_2 + \|(\tilde{Z}_{c_{n,m}}(s))\|_2 \ \|(\tilde{I}_n^{(s)'}(s))\|_2]_{max}$$

<div align="right">(127b)</div>

$$\|((\tilde{v}_n^{(s)}(s))_u)\|_\infty \leq L[\ \|(\tilde{v}_n^{(s)'}(z,s))\|_\infty + \|(\tilde{Z}_{c_{n,m}}(s))\|_\infty \ \|(\tilde{I}_n^{(s)'}(s))\|_\infty]_{max}$$

<div align="right">(127c)</div>

4.   NORM OF THE MATRIX $[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]^{-1}$

From Equations 108 and 117, the 2 norms of the scattering and propagation matrices are less than or equal to one, and, hence, for $s = j\omega$

$$\|((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})\|_2 \leq \|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2 \|((\tilde{S}_{n,m}(s))_{u,v})\|_2$$

$$\leq 1 \tag{128}$$

Then from Equation A48 one can write

$$\|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]^{-1}\|_2$$

$$\leq \frac{1}{1 - \|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2 \ \|((\tilde{S}_{n,m}(s))_{u,v})\|_2} \tag{129}$$

for $s = j\omega$

Note that since the product of the norms in the denominator of Equation 129 is less than or equal to one, this upper bound cannot be used for calculating upper bound for the norm on the left-hand side of Equation 129, for it gives an infinitely large bound which is not useful. To get a finite bound in Equation 129, tighter bounds for the scattering and the propagation matrices are required. For a homogeneous medium, the norm of the propagation matrix is given by Equation 115 as

$$\|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2 = e^{-\alpha(s)L} \qquad \text{for } s = j\omega \tag{130}$$

<div align="center">49</div>

For a lossless case, Equation 130 reduces to (for $s = j\omega$)

$$\| ((\tilde{\Gamma}_{n,m}(s))_{u,v}) \|_2 = 1 \tag{131}$$

Hence, for a lossless case, from Equation 129, one can write

$$\| [((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]^{-1} \|_2 \leq \frac{1}{1 - \| ((\tilde{S}_{n,m}(s))_{u,v}) \|_2}$$
$$\text{for } s = j\omega \tag{132}$$

The norm of the scattering supermatrix for a uniform section of a multiconductor transmission line can be calculated from the knowledge of the termination impedances considered next.

5. NORM OF THE SCATTERING MATRICES AT TERMINATIONS

The scattering or reflection coefficient matrices $(\tilde{S}_{n,m}(s))_{1,2}$ and $(\tilde{S}_{n,m}(s))_{2,1}$ at the terminations of a uniform multiconductor line are given by Equations 25 and 26 in terms of the characteristic-admittance and termination-admittance matrices as

$$(\tilde{S}_{n,m}(s))_{1,2} = [(\tilde{Y}_{C_{n,m}}(s)) + (\tilde{Y}_{T_{n,m}}(0,s))]^{-1} \cdot [(\tilde{Y}_{C_{n,m}}(s)) - (\tilde{Y}_{T_{n,m}}(0,s))] \tag{133}$$

$$(\tilde{S}_{n,m}(s))_{2,1} = [(\tilde{Y}_{C_{n,m}}(s)) + (\tilde{Y}_{T_{n,m}}(L,s))]^{-1} \cdot [(\tilde{Y}_{C_{n,m}}(s)) - (\tilde{Y}_{T_{n,m}}(L,s))] \tag{134}$$

where $(\tilde{Y}_{T_{n,m}}(0,s))$ and $(\tilde{Y}_{T_{n,m}}(L,s))$ are the termination-admittance matrices at $z = 0$ and $z = L$, respectively. These are related to the termination-impedance matrices by the following relations:

$$(\tilde{Y}_{T_{n,m}}(0,s)) = (\tilde{Z}_{T_{n,m}}(0,s))^{-1} \tag{135}$$

$$(\tilde{Y}_{T_{n,m}}(L,s)) = (\tilde{Z}_{T_{n,m}}(L,s))^{-1} \tag{136}$$

From Equations 133 and 134, using Equation A6, one can write

$$\|(\tilde{S}_{n,m}(s))_{1,2}\| \leq \|[(\tilde{Y}_{C_{n,m}}(s)) + (\tilde{Y}_{T_{n,m}}(0,s))]^{-1}\|$$

$$\cdot \|[(\tilde{Y}_{C_{n,m}}(s)) - (\tilde{Y}_{T_{n,m}}(0,s))]\| \tag{137}$$

$$\|(\tilde{S}_{n,m}(s))_{2,1}\| \leq \|[(\tilde{Y}_{C_{n,m}}(s)) + (\tilde{Y}_{T_{n,m}}(L,s))]^{-1}\|$$

$$\cdot \|[(\tilde{Y}_{C_{n,m}}(s)) - (\tilde{Y}_{T_{n,m}}(L,s))]\| \tag{138}$$

For a short- or open-circuit termination (all termination impedances are zero or infinity), the scattering matrices in Equations 133 and 134 are equal to, respectively, - or + the identity matrix. And since the eigenvalues of the identity matrix are all equal to one, the norm of the reflection-coefficient matrices is exactly equal to one for short-circuit or open-circuit termination and, therefore, these two cases will be excluded and the assumption made that the termination impedances are finite and nonzero.

An estimation of upper bounds for norms of scattering matrices in Equations 137 and 138 is quite difficult without a complete knowledge of the characteristic-admittance and termination-admittance matrices. However, things can be simplified somewhat if one assumes that the termination-admittance matrices are real and diagonal; the real, diagonal matrix implies resistive diagonal loads, that is, there are no loads between conductors and each conductor is terminated to ground in a resistive load. This is not a severe assumption since, in practice, diagonal loads are very common for electronic systems connected by multicᴏꞏ ductor cables. Further, one assumes that the medium is lossless, or the losses are small so that the characteristic-admittance matrix is real.

For diagonal resistive loads, one has

$$(\tilde{Y}_{T_{n,m}}(0,s)) = (G_{T_{n,m}}(0))$$

(139)

$$(\tilde{Y}_{T_{n,m}}(L,s)) = (G_{T_{n,m}}(L))$$

where

$$\left.\begin{array}{l} G_{T_{n,m}}(0) = 0 \\[12pt] G_{T_{n,m}}(L) = 0 \end{array}\right\} \quad \text{if } n \neq m$$

and for $n = m$

$G_{T_{n,n}}(0)$ = the conductance between nth and ground conductors at $z = 0$

$G_{T_{n,n}}(L)$ = the conductance between nth and ground conductors at $z = L$

For a lossless case, the characteristic-admittance matrix is independent of frequency and can be written as

$$(\tilde{Y}_{c_{n,m}}(s)) = (\tilde{Y}_{c_{n,m}})$$

(140)

Substituting Equations 139 and 140 into Equations 137 and 138 gives

$$\|(\tilde{S}_{n,m})_{1,2}\|_2 \leq \|[(\tilde{Y}_{c_{n,m}}) + (G_{T_{n,m}}(0))]^{-1}\|_2 \, \|[(\tilde{Y}_{c_{n,m}}) - (G_{T_{n,m}}(0))]\|_2$$

(141)

$$\|(\tilde{S}_{n,m})_{2,1}\|_2 \leq \|[(\tilde{Y}_{c_{n,m}}) + (G_{T_{n,m}}(L))]^{-1}\|_2 \, \|[(\tilde{Y}_{c_{n,m}}) - (G_{T_{n,m}}(L))]\|_2$$

(142)

Note that if the line is terminated in its characteristic admittance, the scattering matrix is a null matrix and its norm is zero. Since the termination-admittance matrices are diagonal, their norms are simply equal to the largest element, i.e.,

$$\|(G_{T_{n,n}}(^0_L))\|_2 = \max G_{T_{n,n}}(^0_L)$$

(143)

Thus the 2 norm of the termination-admittance matrix is equal to the reciprocal of the smallest value of the terminating resistor. The characteristic-admittance matrix is a diagonally dominant, real symmetric matrix (Ref. 14). The diagonally dominant property is defined as (Ref. 7)

$$|\tilde{Y}_{c_{n,n}}| \geq \sum_{\substack{m \\ n \neq m}} |\tilde{Y}_{c_{n,m}}| \qquad \text{for all } n \qquad (144)$$

Since $(G_{T_{n,m}})$ is positive and diagonal, the matrix sum $[(\tilde{Y}_{c_{n,m}}) + (G_{T_{n,m}})]$ is also diagonally dominant. Then from Equation A37 we can write

$$\|[(\tilde{Y}_{c_{n,m}}) + (G_{T_{n,m}}(_L^0))]^{-1}\|_2 \leq \cfrac{1}{\min_{n} \{|\tilde{Y}_{c_{n,n}} + (G_{T_{n,n}}(_L^0)| - \sum_{\substack{m=1 \\ m \neq n}}^{N} |\tilde{Y}_{c_{n,m}} + (G_{T_{n,m}}(_L^0)|\}}$$

$$(145)$$

An upper bound for the characteristic-admittance matrix can be obtained using Equation A38 as

$$\|(\tilde{Y}_{c_{n,m}})\|_2 \leq \sum_{n} |\tilde{Y}_{c_{n,n}}|$$

$$\leq N \max_{n,m} |\tilde{Y}_{c_{n,m}}| \qquad (146)$$

For a homogeneous case, the characteristic-admittance matrix can be obtained from the per-unit-length inductance matrix using the relation

$$(\tilde{Y}_{c_{n,m}}) = \frac{1}{v}(L_{n,m})^{-1} \qquad (147)$$

where v is the speed of propagation on the transmission line. The self and mutual terms of the inductance matrix for a multiconductor line can be estimated approximately, using the following relations (Ref. 10).

53

$$L_{n,n} = 0.2 \, \ell n[4 \, H_n/d_n] \; \mu H/m$$

$$L_{n,m} = 0.2 \, \ell n[B_{n,m}/D_{n,m}] \; \mu H/m$$

(148)

The parameters in Equation 148 are defined as

d   the diameter of the conductor

H   the distance from a conductor to ground plane

D   the distance between two conductors (between centers)

B   the distance from the conductor to the image of a second

The relations in Equation 148 are valid if the distances between conductors are greater than or equal to 5 times the radius of conductors.

Similarly, using the procedure described above, we can calculate an upper bound for the inverse of the reflection coefficient matrices. From Equations 133 and 134, for diagonal, resistive loads and a lossless case, one can write

$$(\tilde{S}_{n,m}(s))^{-1}_{1,2} = [(\tilde{Y}_{c_{n,m}}) - (G_{T_{n,m}}(0))]^{-1} \cdot [(\tilde{Y}_{c_{n,m}}) + (G_{T_{n,m}}(0))] \qquad (149)$$

$$(\tilde{S}_{n,m}(s))^{-1}_{2,1} = [(\tilde{Y}_{c_{n,m}}) - (G_{T_{n,m}}(L))]^{-1} \cdot [(\tilde{Y}_{c_{n,m}}) + (G_{T_{n,m}}(L))] \qquad (150)$$

Using Equation A6 in Equations 149 and 150, one can write

$$\|(\tilde{S}_{n,m}(s))^{-1}_{1,2}\|_2 \leq \|[(\tilde{Y}_{c_{n,m}}) - (G_{T_{n,m}}(0))]^{-1}\|_2 \cdot \|[(\tilde{Y}_{c_{n,m}}) + (G_{T_{n,m}}(0))]\|_2$$

(151)

$$\|(\tilde{S}_{n,m}(s))^{-1}_{2,1}\|_2 \leq \|[(\tilde{Y}_{c_{n,m}}) - (G_{T_{n,m}}(L))]^{-1}\|_2 \cdot \|[(\tilde{Y}_{c_{n,m}}) + (G_{T_{n,m}}(0))]\|_2$$

(152)

The norms in Equations 151 and 152 can be evaluated for diagonal loads using the relations for the norms of the characteristic-admittance matrix, the load admittance matrix and the matrix $[(\tilde{Y}_{c_{n,m}}) - (G_{T_{n,m}})]^{-1}$.

Having defined norms of the scattering and propagation supermatrices and the source supervector, one can now calculate upper and lower bounds for combined voltage waves, voltages, and currents using the relations derived in Section III.

6. BOUNDS FOR COMBINED VOLTAGES, VOLTAGES, AND CURRENTS

Substituting Equation 129 into 75, one obtains an upper bound for the combined voltage waves leaving junctions (terminations for a uniform section of line) as

$$\|((\tilde{V}_n(0,s))_u)\|_\infty \leq \frac{\sqrt{N_s}\; \|((\tilde{S}_{n,m}(s))_{u,v})\|_2 \|((\tilde{V}_n^{(s)}(s))_u)\|_\infty}{1 - \|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2 \|((\tilde{S}_{n,m}(s))_{u,v})\|_2} \tag{153}$$

where $((\tilde{V}_n(0,s))_u)$, $((\tilde{V}_n^{(s)}(s))_u)$, $((\tilde{\Gamma}_{n,m}(s))_{u,v})$ and $((\tilde{S}_{n,m}(s))_{u,v})$ are given by Equations 91, 104, 102, and 97, respectively. Note that N_s is the dimension of the source supervector and is equal to 2N, where N is the number of conductors in the transmission line.

The ∞ norm of the source supervector is given by Equation 124, and the 2 norm of the propagation supermatrix is given by Equation 115. The calculation of the norm of the scattering supermatrix was discussed in Section IV.5. Note that, for a lossless case, the 2 norm of the propagation supermatrix is exactly equal to one (for $s = j\omega$), and use of the inequality (Equation 108) in Equation 153 gives an infinitely large bound for the combined voltage waves leaving the termination, which is not useful. Therefore, the knowledge of a tighter upper bound on the norm of the scattering supermatrix is essential to obtain a practical bound, and this can be obtained by using relations discussed in Section IV.5. A lower bound for the combined voltage waves leaving terminations is given by Equation 77 as

$$\|((\tilde{V}_n(0,s))_u)\|_\infty$$

$$\geq \frac{\|((\tilde{v}_n^{(s)}(s))_u)\|_\infty}{\sqrt{N_s}\|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]\|_2 \|((\tilde{S}_{n,m}(s))_{u,v})^{-1}\|_2} \tag{154}$$

An upper bound for the norm of the inverse of the scattering supermatrix can be obtained from Equations 151 and 152. Note that, in this case, N_s is the order of the supermatrices in the denominator and is equal to 2N.

Using Equation A6 in 154, we get

$$\|((\tilde{V}_n(0,s))_u)\|_\infty$$

$$\geq \frac{\|((\tilde{v}_n^{(s)}(s))_u)\|_\infty}{\sqrt{N_s}[1 + \|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2 \|((\tilde{S}_{n,m}(s))_{u,v})\|_2] \|((S_{n,m}(s))_{u,v})^{-1}\|_2} \tag{155}$$

Substituting Equations 129 and A6 into Equation 79 gives upper bound for the combined voltage waves arriving at the junctions as

$$\|((\tilde{V}_n(L_u,s))_u)\|_\infty \leq \frac{\sqrt{N_s}\|((\tilde{v}_n^{(s)}(s))_u)\|_\infty}{1 - \|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2 \|((\tilde{S}_{n,m}(s))_{u,v})\|_2} \tag{156}$$

where $((\tilde{V}_n(L_u,s))_u$ is given by Equation 92 and $N_s = 2N$.

A lower bound for the combined voltage waves arriving at the junctions is obtained from Equation 80, using Equation A6 as

$$\|((\tilde{V}_n(L_u,s))_u)\|_\infty \geq \frac{\|((\tilde{v}_n^{(s)}(s))_u)\|_\infty}{\sqrt{N_s}[1 + \|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2 \|((\tilde{S}_{n,m}(s))_{u,v})\|_2]} \tag{157}$$

Similarly, substitution of Equation 129 into Equations 81 and 83 gives an upper bound for voltages and currents at the junctions as

$$\|((\tilde{v}_n^{(0)}(s))_u)\|_\infty$$

$$\leq \frac{1}{2} \frac{\sqrt{N_s}\,[\|((\tilde{S}_{n,m}(s))_{u,v})\|_2 + \|((P_{n,m})_{u,v})\|_2]\|((\tilde{v}_n^{(s)}(s))_u)\|_\infty}{1 - \|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2\|((\tilde{S}_{n,m}(s))_{u,v})\|_2} \qquad (158)$$

$$\|(\tilde{I}_n^{(0)}(s))_u\|_\infty$$

$$\leq \frac{1}{2} \frac{\sqrt{N_s}\,\|((\tilde{Y}_{C_{n,m}}(s))_{u,v})\|_2[\|((\tilde{S}_{n,m}(s))_{u,v})\| + \|((P_{n,m})_{u,v})\|]\,\|((\tilde{v}_n(s))_u)\|_\infty}{1 - \|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2\|((\tilde{S}_{n,m}(s))_{u,v})\|_2} \qquad (159)$$

and a lower bound for voltages and currents is obtained from Equations 82 and 84, using Equation A6 as

$$\|((\tilde{v}_n^{(0)}(s))_u)\|_\infty \geq \frac{1}{2}\|((\tilde{v}_n^{(s)}(s))_u)\|_\infty$$

$$\Big/ \sqrt{N_s}[1 + \|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2\|((\tilde{S}_{n,m}(s))_{u,v})\|_2]\|[((\tilde{S}_{n,m}(s))_{u,v}) + ((P_{n,m})_{u,v})]^{-1}\|_2 \qquad (160)$$

$$\|((\tilde{I}_n^{(0)}(s))_u)\|_\infty \geq \frac{1}{2}\|((\tilde{v}_n^{(s)}(s))_u)\|_\infty$$

$$\Big/ \Big[\sqrt{N_s}\,[1 + \|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2\|((\tilde{S}_{n,m}(s))_{u,v})\|_2]$$

$$\cdot \|[((\tilde{S}_{n,m}(s))_{u,v}) - ((\tilde{P}_{n,m})_{u,v})]^{-1}\|_2\|((\tilde{Z}_{C_{n,m}}(s))_{u,v})\|_2\Big] \qquad (161)$$

Since the permutation supermatrix $((P_{n,m})_{u,v})$ is an orthogonal supermatrix, we have

$$\|((P_{n,m})_{u,v})\| = \|((P_{n,m})_{u,v})^{-1}\| = 1 \qquad (162)$$

where $((P_{n,m})_{u,v})$ is given by the definition in Equation 61 as

$$((P_{n,m})_{u,v}) = \begin{pmatrix} (0_{n,m})_{1,1} & (1_{n,m})_{1,2} \\ (1_{n,m})_{2,1} & (0_{n,m})_{2,2} \end{pmatrix} \qquad (163)$$

Also,

$$[((\tilde{S}_{n,m}(s))_{u,v}) + ((P_{n,m})_{u,v})]^{-1}$$

$$= [((1_{n,m})_{u,v}) + ((P_{n,m})_{u,v})^{-1} : ((\tilde{S}_{n,m}(s))_{u,v})]^{-1} : ((P_{n,m})_{u,v})^{-1}$$

and
$$\qquad \qquad \qquad (164)$$

$$\|((P_{n,m})_{u,v})^{-1} : ((\tilde{S}_{n,m}(s))_{u,v})\|_2 \leq \|((\tilde{P}_{n,m})_{u,v})^{-1}\|_2 \|((\tilde{S}_{n,m}(s))_{u,v})\|_2$$
$$\qquad \qquad \qquad (165)$$

Substituting Equations 162 and 108 into Equation 165 gives

$$\|((P_{n,m})_{u,v})^{-1} : ((\tilde{S}_{n,m}(s))_{u,v})\|_2 \leq 1 \qquad \text{for } s = j\omega \qquad (166)$$

Using Equations A47, 162, 166, and A6 in Equation 164, one obtains

$$\|[((\tilde{S}_{n,m}(s))_{u,v}) + ((P_{n,m})_{u,v})]^{-1}\|_2 \leq \frac{1}{1 - \|((P_{n,m})_{u,v})^{-1} : ((S_{n,m}(s))_{u,v})\|_2}$$
$$\text{for } s = j\omega \qquad (167)$$

Similarly, one can write

$$\|[((\tilde{S}_{n,m}(s))_{u,v}) - ((P_{n,m})_{u,v})]^{-1}\|_2 \leq \frac{1}{1 - \|((P_{n,m})_{u,v})^{-1} : ((\tilde{S}_{n,m}(s))_{u,v})\|_2}$$
$$\text{for } s = j\omega \qquad (168)$$

53

Substituting Equation 162 into Equations 158 and 159, and Equations 167 and 168 into Equations 160 and 161, one obtains upper and lower bounds for voltages and currents at the terminations, which are given (for $s = j\omega$) by

$$\|((\tilde{v}_n^{(0)}(s))_u)\|_\infty \leq \frac{1}{2} \frac{\sqrt{N_s}[1 + \|((\tilde{S}_{n,m}(s))_{u,v})\|_2]\|((\tilde{v}_n^{(s)}(s))_u)\|_\infty}{1 - \|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2\|((\tilde{S}_{n,m}(s))_{u,v})\|_2} \qquad (169)$$

$$\|((\tilde{I}_n^{(0)}(s))_u)\|_\infty$$

$$\leq \frac{1}{2} \frac{\sqrt{N_s} \|((\tilde{Y}_{c_{n,m}}(s))_{u,v})\|_2 [1 + \|((\tilde{S}_{n,m}(s))_{u,v})\|_2]\|((\tilde{v}_n^{(s)}(s))_u)\|_\infty}{1 - \|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2\|((\tilde{S}_{n,m}(s))_{u,v})\|_2} \qquad (170)$$

$$\|((\tilde{v}_n^{(0)}(s))_u)\|_\infty$$

$$\geq \frac{1}{2} \frac{\|((\tilde{v}_n^{(s)}(s))_u)\|_\infty[1 - \|((P_{n,m})_{u,v})^{-1} : ((S_{n,m}(s))_{u,v})\|_2]}{\sqrt{N_s}[1 + \|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2\|((\tilde{S}_{n,m}(s))_{u,v})\|_2]} \qquad (171)$$

$$\|((\tilde{I}_n^{(0)}(s))_u)\|_\infty$$

$$\geq \frac{1}{2} \frac{\|((\tilde{v}_n^{(s)}(s))_u)\|_\infty[1 - \|((P_{n,m})_{u,v})^{-1} : ((\tilde{S}_{n,m}(s))_{u,v})\|_2]}{\sqrt{N_s}[1 + \|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2\|((\tilde{S}_{n,m}(s))_{u,v})\|_2]\|((\tilde{Z}_{c_{n,m}}(s))_{u,v})\|_2}$$

$$\qquad (172)$$

From Equation A11, one can write

$$\|((P_{n,m})_{u,v})^{-1} : ((\tilde{S}_{n,m}(s))_{u,v}\|_2$$

$$= [\lambda_{max}\{((\tilde{S}_{n,m}(s))_{u,v})^{\dagger} : ((P_{n,m})_{u,v}) : ((P_{n,m})_{u,v})^{-1} : ((\tilde{S}_{n,m}(s))_{u,v})\}]^{\frac{1}{2}}$$

$$= [\lambda_{max}\{((\tilde{S}_{n,m}(s))_{u,v})^{\dagger} : ((\tilde{S}_{n,m}(s))_{u,v})\}]^{\frac{1}{2}}$$

$$= \|((\tilde{S}_{n,m}(s))_{u,v}\|_2 \tag{173}$$

Substituting Equation 173 into Equations 171 and 172 gives

$$\|((\tilde{V}_n^{(0)}(s))_u)\|_\infty \geq \frac{1}{2} \frac{\|((\tilde{V}_n^{(s)}(s))_u)\|_\infty [1 - \|((\tilde{S}_{n,m}(s))_{u,v})\|_2]}{\sqrt{N_s}[1 + \|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2 \ \|((\tilde{S}_{n,m}(s))_{u,v})\|_2]} \tag{174}$$

$$\|((\tilde{I}_n^{(0)}(s))_u)\|_\infty \geq \frac{1}{2} \frac{\|((\tilde{V}_n^{(s)}(s))_u)\|_\infty [1 - \|((\tilde{S}_{n,m}(s))_{u,v})\|]_2}{\sqrt{N_s}[1 + \|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2 \|((\tilde{S}_{n,m}(s))_{u,v})\|_2] \|((\tilde{Z}_{c_{n,m}}(s))_{u,v})\|_2} \tag{175}$$

The characteristic-impedance supermatrix for the uniform section of a line in Equation 175 is given by the relation

$$((\tilde{Z}_{c_{n,m}}(s))_{u,v}) = \begin{pmatrix} (\tilde{Z}_{c_{n,m}}(s))_{1,1} & (0_{n,m})_{1,2} \\ (0_{n,m})_{2,1} & (\tilde{Z}_{c_{n,m}}(s))_{2,2} \end{pmatrix} \tag{176}$$

and the characteristic-admittance supermatrix is given by the relation

$$((\tilde{Y}_{c_{n,m}}(s))_{u,v}) = ((\tilde{Z}_{c_{n,m}}(s))_{u,v})^{-1}$$

$$= \begin{pmatrix} (\tilde{Y}_{c_{n,m}}(s))_{1,1} & (0_{n,m})_{1,2} \\ (0_{n,m})_{2,1} & (\tilde{Y}_{c_{n,m}}(s))_{2,2} \end{pmatrix} \tag{177}$$

60

where $(\tilde{Z}_{c_{n,m}}(s))$ and $(\tilde{Y}_{c_{n,m}}(s))$ are the characteristic-impedance and admittance matrices of the line.

Thus a lower and an upper bound for the combined voltage waves leaving junctions, the combined voltage waves entering junctions, voltages at the junctions, and currents at the junctions can be calculated using Equations 153, 155; 156, 157; 169, 174; and 170, 175, respectively.

V. BOUNDS FOR A MULTICONDUCTOR TRANSMISSION LINE WITH A BRANCH

Having defined upper and lower bounds for voltages and currents at terminations of a uniform section of a multiconductor transmission line, now consider a somewhat more complicated transmission-line network, a multiconductor line with a branch (T-network). All the branches of the T-network are terminated at their respective ends. Two types of excitations will be considered. In the first type of excitation, the network is excited by an incident external field, and in the second type, the line is excited by voltage or current sources at the terminations.

Consider a multiconductor line T-network as shown in Figure 4. The network topology involves three sections of uniform multiconductor transmission lines (tubes), and four junctions denoted by 1, 2, 3, and 4. The three tubes of the network meet at junction 2. The transmission lines are terminated at their respective ends. Let the number of conductors in tubes 1, 2, and 3 be n_1, n_2, and n_3, respectively, and their lengths be denoted by ℓ_1, ℓ_2, and ℓ_3, respectively. The medium surrounding the network is assumed to be homogeneous. It is assumed that the junction 2 is to be of zero length, and there is no direct coupling between branches. The forward and backward traveling waves on tube 1, tube 2, and tube 3 are denoted by W_1 and W_2, W_3 and W_4, and W_5 and W_6, respectively. The combined voltage vectors at different junctions for various tubes are defined as:

$$
\text{Tube 1}
\begin{cases}
\text{Junction 1}
\begin{cases}
(V_n(0,s))_1 & \text{wave leaving the junction} \\[2ex]
(V_n(L_2,s))_2 & \text{wave arriving at the junction}
\end{cases} \\[4ex]
\text{Junction 2}
\begin{cases}
(\tilde{V}_n(0,s))_2 & \text{wave leaving the junction} \\[2ex]
(\tilde{V}_n(L_1,s))_1 & \text{wave arriving at the junction}
\end{cases}
\end{cases}
\tag{178}
$$
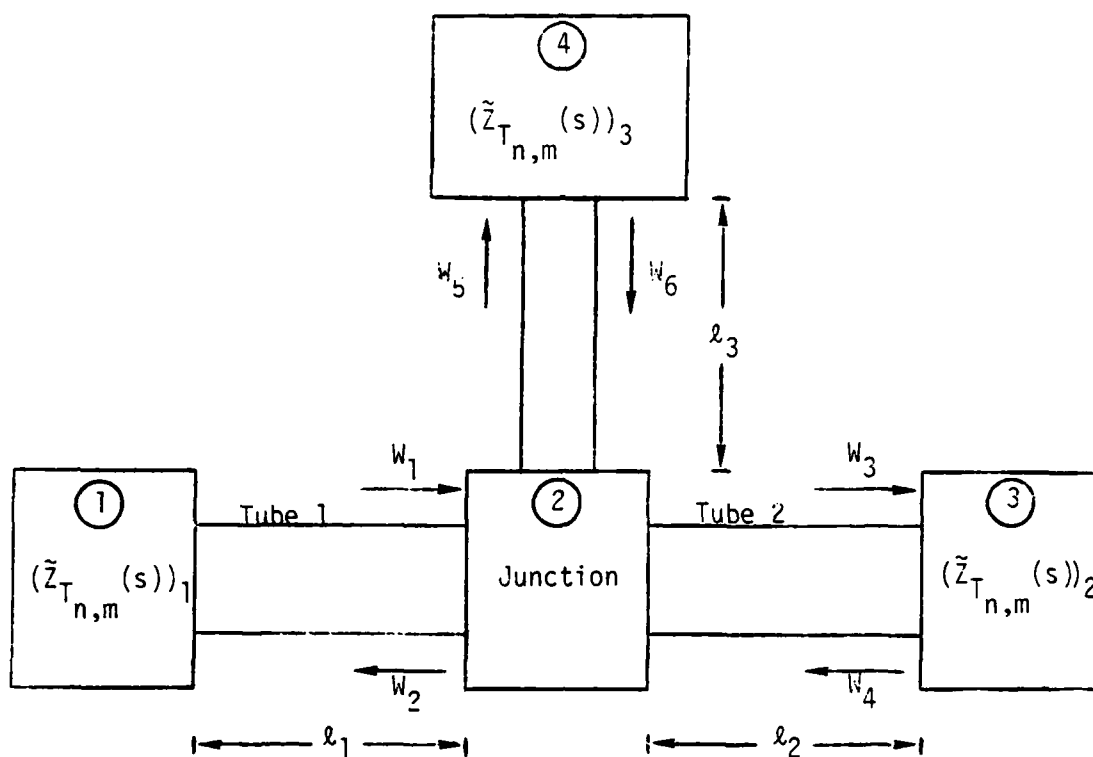
Figure 4. A multiconductor transmission line with a branch.

$$
\text{Tube 2} \quad
\begin{cases}
\text{Junction 2}
\begin{cases}
(\tilde{V}_n(0,s))_3 & \text{wave leaving the junction} \\[2mm]
(\tilde{V}_n(L_4,s))_4 & \text{wave arriving at the junction}
\end{cases} \\[6mm]
\text{Junction 3}
\begin{cases}
(\tilde{V}_n(0,s))_4 & \text{wave leaving the junction} \\[2mm]
(\tilde{V}_n(L_3,s))_3 & \text{wave arriving at the junction}
\end{cases}
\end{cases}
\qquad (179)
$$

$$
\text{Tube 3} \quad
\begin{cases}
\text{Junction 2}
\begin{cases}
(\tilde{V}_n(0,s))_5 & \text{wave leaving the junction} \\[2mm]
(\tilde{V}_n(L_6,s))_6 & \text{wave arriving at the junction}
\end{cases} \\[6mm]
\text{Junction 4}
\begin{cases}
(\tilde{V}_n(0,s))_6 & \text{wave leaving the junction} \\[2mm]
(\tilde{V}_n(L_5,s))_5 & \text{wave arriving at the junction}
\end{cases}
\end{cases}
\qquad (180)
$$

where $L_1 = L_2 = \ell_1$, $L_3 = L_4 = \ell_2$, and $L_5 = L_6 = \ell_3$. Having defined the combined voltage waves for different tubes, one can now define the propagation supermatrix, scattering supermatrix, and the source supermatrix for the network.

1. PROPAGATION SUPERMATRIX

For tube 1, the waves leaving and entering junctions are related through the propagation supermatrix as

$$\begin{pmatrix} (\tilde{V}_n(L_1,s))_1 \\ (\tilde{V}_n(L_2,s))_2 \end{pmatrix} = \begin{pmatrix} (\tilde{\Gamma}_{n,m}(s))_{1,1} & (0_{n,m}) \\ (0_{n,m}) & (\tilde{\Gamma}_{n,m}(s))_{2,2} \end{pmatrix} : \begin{pmatrix} (\tilde{V}_n(0,s))_1 \\ (\tilde{V}_n(0,s))_2 \end{pmatrix}$$

$$+ \begin{pmatrix} \int_0^{L_1} \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))_1[L_1-z']\} \cdot (\tilde{v}_n^{(s)'}(z',s))_1 dz' \\ -\int_0^{L_2} \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))_2 z''\} \cdot (\tilde{v}_n^{(s)'}(z'',s))_2 dz'' \end{pmatrix} \quad (181)$$

where

$$z'' = L_2 - z'$$

$$(\tilde{\Gamma}_{n,m}(s))_{1,1} = (\tilde{\Gamma}_{n,m}(s))_{2,2} = \exp\{-(\tilde{\gamma}_{1c_{n,m}}(s))\ell_1\}$$

$$(\tilde{\gamma}_{c_{n,m}}(s))_1 = (\tilde{\gamma}_{c_{n,m}}(s))_2 = (\tilde{\gamma}_{1c_{n,m}}(s))$$

$$(\tilde{\gamma}_{1c_{n,m}}(s)) \equiv \text{characteristic-propagation matrix for tube 1}$$

Similarly, relations between waves leaving and entering junctions for tubes 2 and 3 can be written as

64

$$\begin{pmatrix} (\tilde{V}_n(L_3,s))_3 \\ (\tilde{V}_n(L_4,s))_4 \end{pmatrix} = \begin{pmatrix} (\tilde{\Gamma}_{n,m}(s))_{3,3} & (0_{n,m}) \\ (0_{n,m}) & (\tilde{\Gamma}_{n,m}(s))_{4,4} \end{pmatrix} : \begin{pmatrix} (\tilde{V}_n(0,s))_3 \\ (\tilde{V}_n(0,s))_4 \end{pmatrix}$$

$$+ \begin{pmatrix} \int_0^{L_3} \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))_3[L_3-z']\} \cdot (\tilde{v}_n^{(s)'}(z',s))_3 dz' \\ -\int_0^{L_4} \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))_4 z''\} \cdot (\tilde{v}_n^{(s)'}(z'',s))_4 dz'' \end{pmatrix} \quad (182)$$

where

$$(\tilde{\Gamma}_{n,m}(s))_{3,3} = (\tilde{\Gamma}_{n,m}(s))_{4,4} = \exp\{-(\tilde{\gamma}_{2c_{n,m}}(s))\ell_2\}$$

$$(\tilde{\gamma}_{c_{n,m}}(s))_3 = (\tilde{\gamma}_{c_{n,m}}(s))_4 = (\tilde{\gamma}_{2c_{n,m}}(s))$$

$$(\tilde{\gamma}_{2c_{n,m}}(s)) \equiv \text{characteristic-propagation matrix for tube 2}$$

$$z'' = L_4 - z'$$

and

$$\begin{pmatrix} (\tilde{V}_n(L_5,s))_5 \\ (\tilde{V}_n(L_6,s))_6 \end{pmatrix} = \begin{pmatrix} (\tilde{\Gamma}_{n,m}(s))_{5,5} & (0_{n,m}) \\ (0_{n,m}) & (\tilde{\Gamma}_{n,m}(s))_{6,6} \end{pmatrix} : \begin{pmatrix} (\tilde{V}_n(0,s))_5 \\ (\tilde{V}_n(0,s))_6 \end{pmatrix}$$

$$+ \begin{pmatrix} \int_0^{L_5} \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))_5[L_5-z']\} \cdot (\tilde{v}_n^{(s)'}(z',s))_5 dz' \\ -\int_0^{L_6} \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))_6 z''\} \cdot (\tilde{v}_n^{(s)'}(z'',s))_6 dz'' \end{pmatrix} \quad (183)$$

65

where

$$(\tilde{\Gamma}_{n,m}(s))_{5,5} = (\tilde{\Gamma}_{n,m}(s))_{6,6} = \exp\{-(\tilde{\gamma}_{3c_{n,m}}(s))\ell_3\}$$

$$(\tilde{\gamma}_{c_{n,m}}(s))_5 = (\tilde{\gamma}_{c_{n,m}}(s))_6 = (\tilde{\gamma}_{3c_{n,m}}(s))$$

$$(\tilde{\gamma}_{3c_{n,m}}(s)) \equiv \text{characteristic-propagation matrix for tube 3}$$
$$z'' = L_6 - z'$$

Note that $(\tilde{\Gamma}_{n,m}(s))_{1,1}$, $(\tilde{\Gamma}_{n,m}(s))_{3,3}$ and $(\tilde{\Gamma}_{n,m}(s))_{5,5}$ are $n_1 \times n_1$, $n_2 \times n_2$, and $n_3 \times n_3$ matrices, respectively, and $(\tilde{\gamma}_{1c_{n,m}}(s))$, $(\tilde{\gamma}_{2c_{n,m}}(s))$, and $(\tilde{\gamma}_{3c_{n,m}}(s))$ are $n_1 \times n_1$, $n_2 \times n_2$ and $n_3 \times n_3$ matrices, respectively.

The network propagation supermatrix $((\tilde{\Gamma}_{n,m}(s))_{u,v})$ and the network source supervector can be obtained by combining the results above in the following manner:

$$
\begin{pmatrix}
(\tilde{V}_n(L_1,s))_1 \\
(\tilde{V}_n(L_2,s))_2 \\
(\tilde{V}_n(L_3,s))_3 \\
(\tilde{V}_n(L_4,s))_4 \\
(\tilde{V}_n(L_5,s))_5 \\
(\tilde{V}_n(L_6,s))_6
\end{pmatrix}
$$

$$
= \begin{pmatrix}
(\tilde{\Gamma}_{n,m}(s))_{1,1} & (0_{n,m}) & (0_{n,m}) & (0_{n,m}) & (0_{n,m}) & (0_{n,m}) \\
(0_{n,m}) & (\tilde{\Gamma}_{n,m}(s))_{2,2} & (0_{n,m}) & (0_{n,m}) & (0_{n,m}) & (0_{n,m}) \\
(0_{n,m}) & (0_{n,m}) & (\tilde{\Gamma}_{n,m}(s))_{3,3} & (0_{n,m}) & (0_{n,m}) & (0_{n,m}) \\
(0_{n,m}) & (0_{n,m}) & (0_{n,m}) & (\tilde{\Gamma}_{n,m}(s))_{4,4} & (0_{n,m}) & (0_{n,m}) \\
(0_{n,m}) & (0_{n,m}) & (0_{n,m}) & (0_{n,m}) & (\tilde{\Gamma}_{n,m}(s))_{5,5} & (0_{n,m}) \\
(0_{n,m}) & (0_{n,m}) & (0_{n,m}) & (0_{n,m}) & (0_{n,m}) & (\tilde{\Gamma}_{n,m}(s))_{6,6}
\end{pmatrix}
$$

$$
\vdots \begin{pmatrix}
(\tilde{V}_n(0,s))_1 \\
(\tilde{V}_n(0,s))_2 \\
(\tilde{V}_n(0,s))_3 \\
(\tilde{V}_n(0,s))_4 \\
(\tilde{V}_n(0,s))_5 \\
(\tilde{V}_n(0,s))_6
\end{pmatrix}
+ \begin{pmatrix}
(\tilde{v}_n^{(s)}(s))_1 \\
(\tilde{v}_n^{(s)}(s))_2 \\
(\tilde{v}_n^{(s)}(s))_3 \\
(\tilde{v}_n^{(s)}(s))_4 \\
(\tilde{v}_n^{(s)}(s))_5 \\
(\tilde{v}_n^{(s)}(s))_6
\end{pmatrix}
\qquad (184)
$$

Equation 184 can be written in supermatrix notation as

$$
((\tilde{V}_n(L_u,s))_u) = ((\tilde{\Gamma}_{n,m}(s))_{u,v}) \; \vdots \; (\tilde{V}_n(0,s))_u) + ((\tilde{v}_n^{(s)}(s))_u) \qquad (185)
$$

where

$$((\tilde{\Gamma}_{n,m}(s))_{u,v}) = (\tilde{\Gamma}_{n,m}(s))_{1,1} \oplus (\tilde{\Gamma}_{n,m}(s))_{2,2} \oplus (\tilde{\Gamma}_{n,m}(s))_{3,3}$$

$$\oplus (\tilde{\Gamma}_{n,m}(s))_{4,4} \oplus (\tilde{\Gamma}_{n,m}(s))_{5,5} \oplus (\tilde{\Gamma}_{n,m}(s))_{6,6} \quad (186)$$

$$\equiv \text{propagation supermatrix}$$

and

$$((\tilde{v}_n^{(s)}(s))_u) = \begin{pmatrix} \int_0^{L_1} \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))_1[L_1-z']\} \cdot (\tilde{v}_n^{(s)'}(z',s))_1 dz' \\[2mm] -\int_0^{L_2} \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))_2 z''\} \cdot (\tilde{v}_n^{(s)'}(z'',s))_2 dz'' \\[2mm] \int_0^{L_3} \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))_3[L_3-z']\} \cdot (\tilde{v}_n^{(s)'}(z',s))_3 dz' \\[2mm] -\int_0^{L_4} \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))_4 z''\} \cdot (\tilde{v}_n^{(s)'}(z'',s))_4 dz'' \\[2mm] \int_0^{L_5} \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))_5[L_5-z']\} \cdot (\tilde{v}_n^{(s)'}(z',s))_5 dz' \\[2mm] -\int_0^{L_6} \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))_6 z''\} \cdot (\tilde{v}_n^{(s)'}(z'',s))_6 dz'' \end{pmatrix}$$

$$\equiv \text{source supervector}$$
$$z'' = L_u - z' \quad \text{for } u = 2,4,6 \quad (187)$$

2. SCATTERING SUPERMATRIX

For convenience in referencing junctions, they are assigned numbers 1, 2, 3, and 4, as shown in Figure 4. For junction 1 where tube 1 is terminated in the impedance $(\tilde{Z}_{T_{n,m}}(s))_1$, the incoming and outgoing waves are related by the following relation:

$$(\tilde{V}_n(0,s))_1 = (\tilde{S}_{n,m}(s))_{1,2} \cdot (\tilde{V}_n(L_1,s))_2 \quad (188)$$

68

From Equation 26, the scattering matrix $(\tilde{S}_{n,m}(s))_{1,2}$ is given by

$$(\tilde{S}_{n,m}(s))_{1,2} = [(\tilde{Y}_{c_{n,m}}(s))_1 + (\tilde{Y}_{T_{n,m}}(s))_1]^{-1} \cdot [(\tilde{Y}_{c_{n,m}}(s))_1 - (\tilde{Y}_{T_{n,m}}(s))_1]$$

(189)

where

$$(\tilde{Y}_{T_{n,m}}(s))_1 = (\tilde{Z}_{T_{n,m}}(s))_1^{-1}$$

and

$$(\tilde{Y}_{c_{n,m}}(s))_1 \equiv \text{characteristic-admittance matrix of tube 1}$$

Similarly, we can write relationships between outgoing and incoming waves at junction 3 and 4 as

$$(\tilde{V}_n(0,s))_4 = (\tilde{S}_{n,m}(s))_{4,3} \cdot (\tilde{V}_n(L_3,s))_3$$

(190)

$$(\tilde{V}_n(0,s))_6 = (\tilde{S}_{n,m}(s))_{6,5} \cdot (\tilde{V}_n(L_5,s))_5$$

(191)

$$(\tilde{S}_{n,m}(s))_{4,3} = [(\tilde{Y}_{c_{n,m}}(s))_2 + (\tilde{Y}_{T_{n,m}}(s))_2]^{-1} \cdot [(\tilde{Y}_{c_{n,m}}(s))_2 - (\tilde{Y}_{T_{n,m}}(s))_2]$$

(192)

$$(\tilde{S}_{n,m}(s))_{6,5} = [(\tilde{Y}_{c_{n,m}}(s))_3 + (\tilde{Y}_{T_{n,m}}(s))_3]^{-1} \cdot [(\tilde{Y}_{c_{n,m}}(s))_3 - (\tilde{Y}_{T_{n,m}}(s))_3]$$

(193)

$$(\tilde{Y}_{T_{n,m}}(s))_2 = (\tilde{Z}_{T_{n,m}}(s))_2^{-1}$$

$$(\tilde{Y}_{T_{n,m}}(s))_3 = (\tilde{Z}_{T_{n,m}}(s))_3^{-1}$$

$$(\tilde{Y}_{c_{n,m}}(s))_2 \equiv \text{characteristic-admittance matrix of tube 2}$$

$$(\tilde{Y}_{c_{n,m}}(s))_3 \equiv \text{characteristic-admittance matrix of tube 3}$$

The outgoing and incoming waves at junction 2 are related in the following manner:

69

$$\begin{pmatrix} (\tilde{V}_n(0,s))_2 \\ (\tilde{V}_n(0,s))_3 \\ (\tilde{V}_n(0,s))_5 \end{pmatrix} = ((\tilde{S}_{n,m}(s))_{u,v})_2 : \begin{pmatrix} (\tilde{V}_n(L_1,s))_1 \\ (\tilde{V}_n(L_4,s))_4 \\ (\tilde{V}_n(L_6,s))_6 \end{pmatrix} \tag{194}$$

where $((\tilde{S}_{n,m}(s))_{u,v})_2$ is the scattering supermatrix of junction 2.

It is assumed here that junction 2 contains only wires which are inter-connected; that is, there are no impedances involved at the junction 2. The procedure for calculating junction scattering supermatrices in general are discussed in References 11 and 12. Here the procedure is illustrated for the case shown in Figure 4. Further, the junction is considered lossless, i.e., all the energy incident at the junction is reflected and/or transmitted.

At a junction where there are several tubes interconnected to one another, Kirchhoff's current law and Kirchhoff's voltage law have to be enforced.

Kirchhoff's current law states that the sum of the currents flowing into a node is zero. For the case where n_1th wire of tube 1 is connected to the n_2th wire of tube 2, and to the n_3th wire of tube 3, and these wires are not connected to any other wires at this junction, one has

$$(I_{n_1}^{(0)}(s))_{r,1} + (I_{n_2}^{(0)}(s))_{r,2} + (I_{n_3}^{(0)}(s))_{r,3} = 0 \tag{195}$$

Equation 195 can be put into supermatrix form, i.e.,

$$\begin{array}{ccc} \text{tube 1} & \text{tube 2} & \text{tube 3} \end{array}$$
$$(0\ 0 \ldots 1 \ldots 0 \mid 0 \ldots 1 \ldots 0 \mid 0\ 0 \ldots 1 \ldots 1 \mid) : \begin{pmatrix} (\tilde{I}_n^{(0)}(s))_{r,1} \\ (\tilde{I}_n^{(0)}(s))_{r,2} \\ (\tilde{I}_n^{(0)}(s))_{r,3} \end{pmatrix} = ((0_n)_r) \tag{196}$$

where $(\tilde{I}_n^{(0)}(s))_{r,1}$, $(\tilde{I}_n^{(0)}(s))_{r,2}$, and $(\tilde{I}_n^{(0)}(s))_{r,3}$ are current vectors at the junction associated with tubes 1, 2, and 3, respectively.

70

In Equation 196, all elements in the left vector are zero, unless they correspond to the conductors which are connected at the node. For N_c connections at the junction, there are N_c equations similar to Equation 196 and we can define the junction connection supermatrix $((C_{I_{n,m}})_{a,b})$ so that

$$((C_{I_{n,m}})_{a,b}) : \begin{pmatrix} (\tilde{I}_n^{(0)}(s))_{r,1} \\ (\tilde{I}_n^{(0)}(s))_{r,2} \\ (\tilde{I}_n^{(0)}(s))_{r,3} \end{pmatrix} = ((0_n)_a) \qquad (197)$$

where $((C_{I_{n,m}})_{a,b})$ is an $N_c \times M_j$ supermatrix, and M_j is the total number of conductors entering the junction. In this case, $M_j = n_1 + n_2 + n_3$.

Kirchhoff's voltage law requires all voltages associated with each conductor to be the same at the same node. Thus for the above example, we have

$$\tilde{V}_{n_1}^{(0)}(s)_{r,1} - \tilde{V}_{n_2}^{(0)}(s)_{r,2} = 0$$
$$\tilde{V}_{n_1}^{(0)}(s)_{r,1} - \tilde{V}_{n_3}^{(0)}(s)_{r,3} = 0 \qquad (198)$$

If there are M conductors being connected to the same node, there are M-1 equations in Equation 198. Equation 198 can also be written in supermatrix form as

$$\begin{pmatrix} 0\ 0...1...0 & 0...-1....0 & 0\ 0.....0...0 \\ 0\ 0...0...0 & 0...\ 1..0.. & 0\ 0..-1..0..0 \\ 0\ 0...0...0 & 0\ 1...0...0 & 0\ 0...-1....0 \end{pmatrix} : \begin{pmatrix} (\tilde{V}_n^{(0)}(s))_{r,1} \\ (\tilde{V}_n^{(0)}(s))_{r,2} \\ (\tilde{V}_n^{(0)}(s))_{r,3} \end{pmatrix} = ((0_n)_a)$$
$$(199)$$

where $(\tilde{V}_n^{(0)}(s))_{r,1}$, $(\tilde{V}_n^{(0)}(s))_{r,2}$, and $(\tilde{V}_n^{(0)}(s))_{r,3}$ are voltage vectors at the junctions associated with tubes 1, 2, and 3, respectively. Here, each row

contains one 1 and one -1, and all other values are zero. Note that the subscripts 1, 2, and 3 on voltage and current vectors denote tube numbers, not waves.

For N_c connections there are $M_j - N_c$ equations. Let us denote the corresponding supermatrix as $((C_{V_{n,m}})_{a,b})$

$$((C_{V_{n,m}})_{a,b}) : \begin{pmatrix} (\tilde{V}_n^{(0)}(s))_{r,1} \\ (\tilde{V}_n^{(0)}(s))_{r,2} \\ (\tilde{V}_n^{(0)}(s))_{r,3} \end{pmatrix} = ((0_n)_r) \qquad (200)$$

At the junction, the total voltage and current are related to the incident and reflected voltage waves as

$$((\tilde{V}_n^{(0)}(s))_r) = \frac{1}{2} [((\tilde{V}_n(s))_r)_+ + ((\tilde{V}_n(s))_r)_-] \qquad (201)$$

$$((\tilde{I}_n^{(0)}(s))_r) = \frac{1}{2} ((\tilde{Y}_{c_{n,m}}(s))_{r,r'})_2 \cdot [((\tilde{V}_n(s))_r)_+ - ((\tilde{V}_n(s))_r)_-] \qquad (202)$$

where $(\tilde{V}_n(s))_{r;+}$ and $(\tilde{V}_n(s))_{r;-}$ are outgoing and incoming waves on the rth tube in the form of combined voltage vectors at the junction, and $((\tilde{Y}_{c_{n,m}}(s))_{r,r'})_2$ is the characteristic-admittance matrix of the junction and is given by Equation 31 as

$$((\tilde{Y}_{c_{n,m}}(s))_{r,r'})_2 = \bigoplus_{r=1}^{3} (\tilde{Y}_{c_{n,m}}(s))_{r,r;2} \qquad (203)$$

where $(\tilde{Y}_{c_{n,m}}(s))_{r,r';\nu}$ is the characteristic-admittance matrix of the rth tube, at junction 2.

Using Equations 197 and 202 gives

72

$$((C_{I_{n,m}})_{a,b}) : ((\tilde{Y}_{c_{n,m}}(s))_{r,r'})_2 : ((\tilde{V}_n(s))_r)_+$$

$$= ((C_{I_{n,m}})_{a,b}) : ((\tilde{Y}_{c_{n,m}}(s))_{r,r'})_2 : ((\tilde{V}_n(s))_r)_- \qquad (204a)$$

Premultiply Equation 204a by a normalizing nonsingular impedance supermatrix $((\tilde{Z}_{n,m}(s))_{a,b})$,

$$((\tilde{Z}_{n,m}(s))_{a,b}) : ((C_{I_{n,m}})_{a,b}) : ((\tilde{Y}_{c_{n,m}}(s))_{r,r'})_2 : ((\tilde{V}_n(s))_r)_+$$

$$= ((\tilde{Z}_{n,m}(s))_{a,b}) : ((C_{I_{n,m}})_{a,b}) : ((\tilde{Y}_{c_{n,m}}(s))_{r,r'}) : ((\tilde{V}_n(s))_r)_-$$

$$\qquad\qquad (204b)$$

$$a,b = 1,2,\ldots,N_c$$

and, similarly, from Equations 200 and 201, one gets

$$(C_{V_{n,m}}) : ((\tilde{V}_n(s))_r)_+ = -(C_{V_{n,m}}) : ((\tilde{V}_n(s))_r)_- \qquad (205)$$

Note that $((C_{I_{n,m}})_{a,b})$ and $((C_{V_{n,m}})_{a,b})$ are supermatrices of size $N_c \times M_j$ and $(M_j - N_c) \times M_j$, while $(\tilde{Y}_{c_{n,m}}(s))_{r,r'})_2$ is of size $M_j \times M_j$. The vectors $((\tilde{V}_n(s))_r)_+$ and $((\tilde{V}_n(s))_r)_-$ are of size M_j.

Combining Equations 204 and 205 gives

$$\begin{pmatrix} -((C_{V_{n,m}})_{a,b}) \\ ((\tilde{Z}_{n,m}(s))_{a,b}) : ((C_{I_{n,m}})_{a,b}) : ((\tilde{Y}_{c_{n,m}}(s))_{r,r'})_2 \end{pmatrix} : ((\tilde{V}_n(s))_r)_+$$

$$= \begin{pmatrix} ((C_{V_{n,m}})_{a,b}) \\ ((\tilde{Z}_{n,m}(s))_{a,b}) : ((C_{I_{n,m}})_{a,b}) : ((\tilde{Y}_{c_{n,m}}(s))_{r,r'})_2 \end{pmatrix} : ((\tilde{V}_n(s))_r)_-$$

$$\qquad\qquad (206)$$

or

73

$$((\tilde{V}_n(s))_r)_+ = \begin{pmatrix} -((C_{V_{n,m}})_{a,b}) \\ ((\tilde{Z}_{n,m}(s))_{a,b}) \vdots ((C_{I_{n,m}})_{a,b}) \vdots ((\tilde{Y}_{c_{n,m}}(s))_{r,r'})_2 \end{pmatrix}^{-1}$$

$$\vdots \begin{pmatrix} ((C_{V_{n,m}})_{a,b}) \\ ((\tilde{Z}_{n,m}(s))_{a,b}) \vdots ((C_{I_{n,m}})_{a,b}) \vdots ((\tilde{Y}_{c_{n,m}}(s))_{r,r'})_2 \end{pmatrix} \vdots ((\tilde{V}_n(s))_r)_- \qquad (207)$$

From Equation 207, the scattering supermatrix for junction 2 is

$$((\tilde{S}_{n,m}(s))_{r,r'})_2 = \begin{pmatrix} -((C_{V_{n,m}})_{a,b}) \\ ((\tilde{Z}_{n,m}(s))_{a,b}) \vdots ((C_{I_{n,m}})_{a,b}) \vdots ((\tilde{Y}_{c_{n,m}}(s))_{r,r'})_2 \end{pmatrix}^{-1}$$

$$\vdots \begin{pmatrix} ((C_{V_{n,m}})_{a,b}) \\ ((\tilde{Z}_{n,m}(s))_{a,b}) \vdots ((C_{I_{n,m}})_{a,b}) \vdots ((\tilde{Y}_{c_{n,m}}(s))_{r,r'})_2 \end{pmatrix} \qquad (208)$$

Note that the normalizing supermatrix $((\tilde{Z}_{n,m}(s))_{a,b})$ makes the two supermatrices in Equation 208 unitless and well conditioned. Without the supermatrix $((\tilde{Z}_{n,m}(s))_{a,b})$, the elements of matrix $((C_{I_{n,m}})_{a,b}) \vdots ((\tilde{Y}_{c_{n,m}}(s))_{r,r'})_2$ will be small compared to the elements of $((C_{V_{n,m}})_{a,b})$.

For the network of Figure 4, the outgoing and incoming combined voltage waves at junction 2 are given by

$$((\tilde{V}_n(s))_r)_+ = \begin{pmatrix} (\tilde{V}_n(0,s))_2 \\ (\tilde{V}_n(0,s))_3 \\ (\tilde{V}_n(0,s))_5 \end{pmatrix} \tag{209}$$

$$((\tilde{V}_n(s))_r)_- = \begin{pmatrix} (\tilde{V}_n(L_1,s))_1 \\ (\tilde{V}_n(L_4,s))_4 \\ (\tilde{V}_n(L_6,s))_6 \end{pmatrix} \tag{210}$$

The scattering supermatrix in Equation 208 is of the order $M_j \times M_j$.

For convenience in properly ordering variables in the scattering super-matrix for the network, let us write the scattering supermatrix for junction 2 in terms of its block matrices; then using Equations 209 and 210 in Equation 207 gives

$$\begin{pmatrix} (\tilde{V}_n(0,s))_2 \\ (\tilde{V}_n(0,s))_3 \\ (\tilde{V}_n(0,s))_5 \end{pmatrix} = \begin{pmatrix} (\tilde{S}_{n,m}(s))_{2,1} & (\tilde{S}_{n,m}(s))_{2,4} & (\tilde{S}_{n,m}(s))_{2,6} \\ (\tilde{S}_{n,m}(s))_{3,1} & (\tilde{S}_{n,m}(s))_{3,4} & (\tilde{S}_{n,m}(s))_{3,6} \\ (\tilde{S}_{n,m}(s))_{5,1} & (\tilde{S}_{n,m}(s))_{5,4} & (\tilde{S}_{n,m}(s))_{5,6} \end{pmatrix} : \begin{pmatrix} (\tilde{V}_n(L_1,s))_1 \\ (\tilde{V}_n(L_4,s))_4 \\ (\tilde{V}_n(L_6,s))_6 \end{pmatrix}$$

$$\tag{211}$$

Combining Equations 188, 190, 191, and 211, and rearranging the junction scattering matrices so that the ordering of the components of the incident and reflected waves is the same as in the propagation supermatrix equation, one gets

$$
\begin{pmatrix}
(\tilde{V}_n(0,s))_1 \\
(\tilde{V}_n(0,s))_2 \\
(\tilde{V}_n(0,s))_3 \\
(\tilde{V}_n(0,s))_4 \\
(\tilde{V}_n(0,s))_5 \\
(\tilde{V}_n(0,s))_6
\end{pmatrix}
$$

$$
=
\begin{pmatrix}
(0_{n,m}) & (\tilde{S}_{n,m}(s))_{1,2} & (0_{n,m}) & (0_{n,m}) & (0_{n,m}) & (0_{n,m}) \\
(\tilde{S}_{n,m}(s))_{2,1} & (0_{n,m}) & (0_{n,m}) & (\tilde{S}_{n,m}(s))_{2,4} & (0_{n,m}) & (\tilde{S}_{n,m}(s))_{2,6} \\
(\tilde{S}_{n,m}(s))_{3,1} & (0_{n,m}) & (0_{n,m}) & (\tilde{S}_{n,m}(s))_{3,4} & (0_{n,m}) & (\tilde{S}_{n,m}(s))_{3,6} \\
(0_{n,m}) & (0_{n,m}) & (\tilde{S}_{n,m}(s))_{4,3} & (0_{n,m}) & (0_{n,m}) & (0_{n,m}) \\
(\tilde{S}_{n,m}(s))_{5,1} & (0_{n,m}) & (0_{n,m}) & (\tilde{S}_{n,m}(s))_{5,4} & (0_{n,m}) & (\tilde{S}_{n,m}(s))_{5,6} \\
(0_{n,m}) & (0_{n,m}) & (0_{n,m}) & (0_{n,m}) & (\tilde{S}_{n,m}(s))_{6,5} & (0_{n,m})
\end{pmatrix}
$$

$$
\cdot
\begin{pmatrix}
(\tilde{V}_n(L_1,s)_1 \\
(\tilde{V}_n(L_2,s)_2 \\
(\tilde{V}_n(L_3,s)_3 \\
(\tilde{V}_n(L_4,s)_4 \\
(\tilde{V}_n(L_5,s)_5 \\
(\tilde{V}_n(L_6,s)_6
\end{pmatrix}
\tag{212}
$$

From Equation 212, the scattering supermatrix of the network is

$$((\tilde{S}_{n,m}(s))_{u,v})$$

$$= \begin{pmatrix}
(0_{n,m}) & (\tilde{S}_{n,m}(s))_{1,2} & (0_{n,m}) & (0_{n,m}) & (0_{n,m}) & (0_{n,m}) \\
(\tilde{S}_{n,m}(s))_{2,1} & (0_{n,m}) & (0_{n,m}) & (\tilde{S}_{n,m}(s))_{2,4} & (0_{n,m}) & (\tilde{S}_{n,m}(s))_{2,6} \\
(\tilde{S}_{n,m}(s))_{3,1} & (0_{n,m}) & (0_{n,m}) & (\tilde{S}_{n,m}(s))_{3,4} & (0_{n,m}) & (\tilde{S}_{n,m}(s))_{3,6} \\
(0_{n,m}) & (0_{n,m}) & (\tilde{S}_{n,m}(s))_{4,3} & (0_{n,m}) & (0_{n,m}) & (0_{n,m}) \\
(\tilde{S}_{n,m}(s))_{5,1} & (0_{n,m}) & (0_{n,m}) & (\tilde{S}_{n,m}(s))_{5,4} & (0_{n,m}) & (\tilde{S}_{n,m}(s))_{5,6} \\
(0_{n,m}) & (0_{n,m}) & (0_{n,m}) & (0_{n,m}) & (\tilde{S}_{n,m}(s))_{6,5} & (0_{n,m})
\end{pmatrix}$$

(213)

The size of the matrix in Equation 213 is $2N \times 2N$, where $N = n_1 + n_2 + n_3$. Note that all the block matrices on the diagonal are null matrices.

3.   NORM OF THE PROPAGATION SUPERMATRIX

The propagation supermatrix given by Equation 186 is block-diagonal, with block matrices equal to the propagation matrices of the various uniform sections of the line (tubes). From Equations A73 and 186, the 2 norm of the scattering supermatrix can be written as

$$\|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2 = \max_r \|(\tilde{\Gamma}_{n,m}(s))_{r,r}\|_2 \tag{214}$$

where r is the tube number ($r = 1,2,3$).

The 2 norm of the propagation matrix of a uniform section of a multiconductor line was discussed in Section IV.2. For a homogeneous medium surrounding the multiconductor cable network, from Equation 115, we have (for $s = j\omega$)

$$\|(\tilde{\Gamma}_{n,m}(s))_{r,r}\|_2 = \exp\{-\alpha_r(s)\ell_r\} \le 1 \tag{215}$$

Substitution of Equation 215 into 214 yields

$$\|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2 = \max_r[\exp\{-\alpha_r(s)\ell_r\}] \le 1 \qquad (216)$$

where the subscript r represents the tube number.

4.   NORM OF THE SCATTERING SUPERMATRIX

For passive terminations, the 2 norm of the scattering supermatrix satisfies the inequality

$$\|((\tilde{S}_{n,m}(s))_{u,v})\|_2 \le 1 \qquad \text{(for s = j}\omega) \qquad (217)$$

because reflected power from all junctions is always less than or equal to incident power for physically realizable systems (power conservation). The following derivation illustrates the proof for Equation 217.

The power-conservation condition can be expressed for lossless tubes (see Appendix B) as

$$((\tilde{V}_n(L_u,s))_u) : ((\tilde{I}_n(L_u,s))_u)^* \le ((\tilde{V}_n(0,s))_u) : ((\tilde{I}_n(0,s))_u)^* \qquad (218)$$

where $((\tilde{I}_n(0,s))_u)$ and $((\tilde{I}_n(L_u,s))_u)$ are the combined current supervectors for waves leaving and entering junctions, respectively.

The combined current vectors are related to the combined voltage vectors in the following manner:

$$((\tilde{V}_n(0,s))_u) = ((\tilde{Z}_{c_{n,m}}(s))_{u,v}) : ((\tilde{I}_n(0,s))_u) \qquad (219)$$

$$((\tilde{V}_n(L,s))_u) = ((\tilde{Z}_{c_{n,m}}(s))_{u,v}) : ((\tilde{I}_n(L,s))_u) \qquad (220)$$

$$((\tilde{I}_n(0,s))_u) = ((\tilde{Y}_{c_{n,m}}(s))_{u,v}) : ((\tilde{V}_n(0,s))_u) \qquad (221)$$

$$((\tilde{I}_n(L,s))_u) = ((\tilde{Y}_{c_{n,m}}(s))_{u,v}) : ((\tilde{V}_n(L,s))_u) \qquad (222)$$

where

$$((\tilde{Z}_{c_{n,m}}(s))_{u,v}) = ((\tilde{Y}_{c_{n,m}}(s))_{u,v})^{-1}$$

$$\equiv \text{characteristic-impedance matrix of the network} \tag{223}$$

The characteristic-impedance matrix for the network is given by

$$((\tilde{Z}_{c_{n,m}}(s))_{u,v}) =$$

$$\begin{pmatrix} (\tilde{Z}_{c_{n,m}}(s))_{1,1} & (0_{n,m}) & (0_{n,m}) & (0_{n,m}) & (0_{n,m}) & (0_{n,m}) \\ (0_{n,m}) & (\tilde{Z}_{c_{n,m}}(s))_{2,2} & (0_{n,m}) & (0_{n,m}) & (0_{n,m}) & (0_{n,m}) \\ (0_{n,m}) & (0_{n,m}) & (\tilde{Z}_{c_{n,m}}(s))_{3,3} & (0_{n,m}) & (0_{n,m}) & (0_{n,m}) \\ (0_{n,m}) & (0_{n,m}) & (0_{n,m}) & (\tilde{Z}_{c_{n,m}}(s))_{4,4} & (0_{n,m}) & (0_{n,m}) \\ (0_{n,m}) & (0_{n,m}) & (0_{n,m}) & (0_{n,m}) & (\tilde{Z}_{c_{n,m}}(s))_{5,5} & (0_{n,m}) \\ (0_{n,m}) & (0_{n,m}) & (0_{n,m}) & (0_{n,m}) & (0_{n,m}) & (\tilde{Z}_{c_{n,m}}(s))_{6,6} \end{pmatrix}$$

The previous equation can be written as

$$((\tilde{Z}_{c_{n,m}}(s))_{u,v}) = \bigoplus_{u=1}^{6} (\tilde{Z}_{c_{n,m}}(s))_{u,u} \tag{224}$$

where $(\tilde{Z}_{c_{n,m}}(s))_{u,u}$ is the characteristic-impedance matrix of the tube associated with the uth wave. Substitution of Equations 221 and 222 into 218 yields

$$((\tilde{V}_n(L_u,s))_u) : ((\tilde{Y}_{c_{n,m}}(s))_{u,v})^{\dagger} : ((\tilde{V}_n(L_u,s))_u)^{*}$$

$$\leq ((\tilde{V}_n(0,s))_u) : ((\tilde{Y}_{c_{n,m}}(s))_{u,v})^{\dagger} : ((\tilde{V}_n(0,s))_u)^{*} \tag{225}$$

79

Equation 225 is similar to Equation B5 and, following the procedure of Appendix B, one can easily prove that, for diagonal symmetrical loads,

$$\|((\tilde{S}_{n,m}(s))_{u,v})\|_2 \leq 1 \qquad \text{(for } s = j\omega) \qquad (226)$$

and

$$\|((\tilde{S}_{n,m}(s))_{u,v})^{-1}\|_2 \geq 1 \qquad \text{(for } s = j\omega) \qquad (227)$$

a. <u>Norm of the junction scattering supermatrix</u>--The 2 norm of the junction scattering supermatrix given in Equation 211 is less than or equal to 1. This can easily be proven by following the procedure described above for the network scattering supermatrix. The 2 norm of the junction scattering supermatrix of a lossless junction is exactly equal to one. Further, the junction scattering supermatrix of a lossless junction is unitary (Ref. 13). The proof of these properties is illustrated in Appendix C.

b. <u>Norm of the scattering supermatrix in terms of its block matrices</u>-- An upper and lower bound for the 2 norm of the scattering supermatrix can be obtained in terms of the 2 norms of its elementary block matrices using the relation (Eq. A102) in Appendix A. From Equation A102 the 2 norm of the scattering supermatrix is bounded by the following relation:

$$\frac{1}{\sqrt{N_s N_v}} \max_{u,v} \|((\tilde{S}_{n,m})_{u,v})\|_2 \leq ((\tilde{S}_{n,m}(s))_{u,v})\|_2 \leq \sqrt{N_s} \max_v \sum_{u=1}^{N} \sqrt{N_v} \|(\tilde{S}_{n,m})_{u,v})\|_2$$

$$(228)$$

Note that the block matrices in Equation 213 are of two kinds: (1) the reflection coefficient matrices at the terminations, (2) partitioned block matrices of the junction scattering supermatrix. An upper bound for the reflection coefficient matrices can be obtained from the knowledge of the termination

80

impedance and the characteristic-impedance matrices of the tubes, using the relations in Section IV.5. The junction scattering supermatrix is obtained from the knowledge of interconnections at the junction.

In Equation 228, due to the presence of factors N_s and N, the upper and lower bounds for the 2 norm of the scattering supermatrix may be very loose and may not be very practical, since the upper bound for the 2 norm of the scattering supermatrix is one.

5. NORM OF THE SOURCE SUPERVECTOR

The source supervector is given by Equation 187, and using Equation A62 its norm can be expressed as

$$\| ((\tilde{V}_n^{(s)}(s))_u) \| =$$

$$\left\| \begin{pmatrix} \left\| \int_0^{L_1} \exp\{-(\tilde{\gamma}_{1c_{n,m}}(s))[L_1-z']\} \cdot [(\tilde{V}_{1n}^{(s)'}(z',s)) + (\tilde{Z}_{c_{n,m}}(s))_{1,1} \cdot (\tilde{I}_{1n}^{(s)'}(z',s))]dz' \right\| \\ \left\| -\int_0^{L_2} \exp\{-(\tilde{\gamma}_{1c_{n,m}}(s))z''\} \cdot [(\tilde{V}_{1n}^{(s)'}(z'',s)) - (\tilde{Z}_{c_{n,m}}(s))_{1,1} \cdot (\tilde{I}_{1n}^{(s)'}(z'',s))]dz'' \right\| \\ \left\| \int_0^{L_3} \exp\{-(\tilde{\gamma}_{2c_{n,m}}(s))[L_3-z']\} \cdot [(\tilde{V}_{2n}^{(s)'}(z',s)) + (\tilde{Z}_{c_{n,m}}(s))_{2,2} \cdot (\tilde{I}_{2n}^{(s)'}(z',s))]dz' \right\| \\ \left\| -\int_0^{L_4} \exp\{-(\tilde{\gamma}_{2c_{n,m}}(s))z''\} \cdot [(\tilde{V}_{2n}^{(s)'}(z'',s)) - (\tilde{Z}_{c_{n,m}}(s))_{2,2} \cdot (\tilde{I}_{2n}^{(s)'}(z'',s))]dz'' \right\| \\ \left\| \int_0^{L_5} \exp\{-(\tilde{\gamma}_{3c_{n,m}}(s))[L_5-z']\} \cdot [(\tilde{V}_{3n}^{(s)'}(z',s)) + (\tilde{Z}_{c_{n,m}}(s))_{3,3} \cdot (\tilde{I}_{3n}^{(s)'}(z',s))]dz' \right\| \\ \left\| -\int_0^{L_6} \exp\{-(\tilde{\gamma}_{3c_{n,m}}(s))z''\} \cdot [(\tilde{V}_{3n}^{(s)'}(z'',s)) - (\tilde{Z}_{c_{n,m}}(s))_{3,3} \cdot (\tilde{I}_{3n}^{(s)'}(z'',s))]dz'' \right\| \end{pmatrix} \right\|$$

$$z'' = L_u - z' \quad \text{for } u = 2,4,6 \tag{229}$$

where $(\tilde{V}_{rn}^{(s)'}(z',s))$ and $(\tilde{I}_{rn}^{(s)'}(z',s))$ are the per-unit-length voltage and current source vectors, respectively, on the rth tube.

For a homogeneous medium, from Equations 121 and 122, the norms of the exponential matrices in Equation 229 are less than or equal to one. Following the procedure used in the derivation of Equation 123, an upper bound for the norm of the source supervector is obtained as (for $s = j\omega$)

$$\| ((\tilde{v}_n^{(s)}(s))_u) \|$$

$$\leq \left\| \left\| \begin{pmatrix} \int_0^{L_1} [\|(\tilde{v}_{1n}^{(s)\prime}(z',s))\| + \|(\tilde{Z}_{c_{n,m}}(s))_{1,1}\| \; \|(\tilde{I}_{1n}^{(s)\prime}(z',s))\|]dz' \\ \int_0^{L_2} [\|(\tilde{v}_{1n}^{(s)\prime}(z'',s))\| + \|(\tilde{Z}_{c_{n,m}}(s))_{1,1}\| \; \|(\tilde{I}_{1n}^{(s)\prime}(z'',s))\|]dz'' \\ \int_0^{L_3} [\|(\tilde{v}_{2n}^{(s)\prime}(z',s))\| + \|(\tilde{Z}_{c_{n,m}}(s))_{2,2}\| \; \|(\tilde{I}_{2n}^{(s)\prime}(z',s))\|]dz' \\ \int_0^{L_4} [\|(\tilde{v}_{2n}^{(s)\prime}(z'',s))\| + \|(\tilde{Z}_{c_{n,m}}(s))_{2,2}\| \; \|(\tilde{I}_{2n}^{(s)\prime}(z'',s))\|]dz'' \\ \int_0^{L_5} [\|(\tilde{v}_{3n}^{(s)\prime}(z',s))\| + \|(\tilde{Z}_{c_{n,m}}(s))_{3,3}\| \; \|(\tilde{I}_{3n}^{(s)\prime}(z',s))\|]dz' \\ \int_0^{L_6} [\|(\tilde{v}_{3n}^{(s)\prime}(z'',s))\| + \|(\tilde{Z}_{c_{n,m}}(s))_{3,3}\| \; \|(\tilde{I}_{3n}^{(s)\prime}(z'',s))\|]dz'' \end{pmatrix} \right\| \right\|$$

(230)

If the per-unit-length voltage and current source vectors along the tubes can be expressed as delta functions as

$$(\tilde{v}_{r,n}^{(s)\prime}(z',s)) = \sum_{\sigma_r=1}^{\sigma_{r_{max}}} (\tilde{v}_{r,n}^{(s)}(s))_{\sigma_r} \delta(z' - \zeta_{\sigma_r})$$

$$(\tilde{I}_{r,n}^{(s)\prime}(z',s)) = \sum_{\sigma_r=1}^{\sigma_{r_{max}}} (\tilde{I}_{r,n}^{(s)}(s))_{\sigma_r} \delta(z' - \zeta_{\sigma_r})$$

where

$$\sigma_r = 1,2,\ldots,\sigma_{r_{max}}$$

$$r = 1,2,3$$

then Equation 230 can be written as

$$\|(\tilde{v}_n^{(s)}(s))_u\|$$

$$\leq \left(\begin{array}{c} \displaystyle\sum_{\sigma_1=1}^{\sigma_{1max}} [\|(\tilde{v}_{1n}^{(s)}(s))\sigma_1\| + \|(\tilde{Z}_{c_{n,m}}(s))_{1,1}\|\|(\tilde{I}_{1n}^{(s)}(s))\sigma_1\|] \\[1em] \displaystyle\sum_{\sigma_1=1}^{\sigma_{1max}} [\|(\tilde{v}_{1n}^{(s)}(s))\sigma_1\| + \|(\tilde{Z}_{c_{n,m}}(s))_{1,1}\|\|(\tilde{I}_{1n}^{(s)}(s))\sigma_1\|] \\[1em] \displaystyle\sum_{\sigma_2=1}^{\sigma_{2max}} [\|(\tilde{v}_{2n}^{(s)}(s))\sigma_2\| + \|(\tilde{Z}_{c_{n,m}}(s))_{2,2}\|\|(\tilde{I}_{2n}^{(s)}(s))\sigma_2\|] \\[1em] \displaystyle\sum_{\sigma_2=1}^{\sigma_{2max}} [\|(\tilde{v}_{2n}^{(s)}(s))\sigma_2\| + \|(\tilde{Z}_{c_{n,m}}(s))_{2,2}\|\|(\tilde{I}_{2n}^{(s)}(s))\sigma_2\|] \\[1em] \displaystyle\sum_{\sigma_3=1}^{\sigma_{3max}} [\|(\tilde{v}_{3n}^{(s)}(s))\sigma_3\| + \|(\tilde{Z}_{c_{n,m}}(s))_{3,3}\|\|(\tilde{I}_{3n}^{(s)}(s))\sigma_3\|] \\[1em] \displaystyle\sum_{\sigma_3=1}^{\sigma_{3max}} [\|(\tilde{v}_{3n}^{(s)}(s))\sigma_3\| + \|(\tilde{Z}_{c_{n,m}}(s))_{3,3}\|\|(\tilde{I}_{3n}^{(s)}(s))\sigma_3\|] \end{array}\right)$$

(231)

Equation 231 can be simplified for the following three special cases:

a.   Sources are delta functions; that is, the sources exist only at a point along the tubes (localized sources). In this case, Equation 231 reduces to

$$\left\| \left(\tilde{V}_n^{(s)}(s) \right)_u \right\|$$

$$\leq \left\| \left\| \begin{pmatrix} \left\| \left(\tilde{V}_{1n}^{(s)}(z',s) \right) \right\| + \left\| \left(\tilde{Z}_{c_{n,m}}(s) \right)_{1,1} \right\| \ \left\| \left(\tilde{I}_{1n}^{(s)}(z',s) \right) \right\| \\[1ex] \left\| \left(\tilde{V}_{1n}^{(s)}(z'',s) \right) \right\| + \left\| \left(\tilde{Z}_{c_{n,m}}(s) \right)_{1,1} \right\| \ \left\| \left(\tilde{I}_{1n}^{(s)}(z'',s) \right) \right\| \\[1ex] \left\| \left(\tilde{V}_{2n}^{(s)}(z',s) \right) \right\| + \left\| \left(\tilde{Z}_{c_{n,m}}(s) \right)_{2,2} \right\| \ \left\| \left(\tilde{I}_{2n}^{(s)}(z',s) \right) \right\| \\[1ex] \left\| \left(\tilde{V}_{2n}^{(s)}(z'',s) \right) \right\| + \left\| \left(\tilde{Z}_{c_{n,m}}(s) \right)_{2,2} \right\| \ \left\| \left(\tilde{I}_{2n}^{(s)}(z'',s) \right) \right\| \\[1ex] \left\| \left(\tilde{V}_{3n}^{(s)}(z',s) \right) \right\| + \left\| \left(\tilde{Z}_{c_{n,m}}(s) \right)_{3,3} \right\| \ \left\| \left(\tilde{I}_{3n}^{(s)}(z',s) \right) \right\| \\[1ex] \left\| \left(\tilde{V}_{3n}^{(s)}(z'',s) \right) \right\| + \left\| \left(\tilde{Z}_{c_{n,m}}(s) \right)_{3,3} \right\| \ \left\| \left(\tilde{I}_{3n}^{(s)}(z'',s) \right) \right\| \end{pmatrix} \right\| \right\|$$

(232)

b. Sources are uniform along the line. In this case, Equation 231 reduces to

$$\left\| \left(\tilde{V}_n^{(s)}(s) \right)_u \right\|$$

$$\leq \left\| \left\| \begin{pmatrix} \left\| \left(\tilde{V}_{1n}^{(s)'}(s) \right) \right\| \ell_1 + \left\| \left(\tilde{Z}_{c_{n,m}}(s) \right)_{1,1} \right\| \ \left\| \left(\tilde{I}_{1n}^{(s)'}(s) \right) \right\| \ell_1 \\[1ex] \left\| \left(\tilde{V}_{1n}^{(s)'}(s) \right) \right\| \ell_1 + \left\| \left(\tilde{Z}_{c_{n,m}}(s) \right)_{1,1} \right\| \ \left\| \left(\tilde{I}_{1n}^{(s)'}(s) \right) \right\| \ell_1 \\[1ex] \left\| \left(\tilde{V}_{2n}^{(s)'}(s) \right) \right\| \ell_2 + \left\| \left(\tilde{Z}_{c_{n,m}}(s) \right)_{2,2} \right\| \ \left\| \left(\tilde{I}_{2n}^{(s)'}(s) \right) \right\| \ell_2 \\[1ex] \left\| \left(\tilde{V}_{2n}^{(s)'}(s) \right) \right\| \ell_2 + \left\| \left(\tilde{Z}_{c_{n,m}}(s) \right)_{2,2} \right\| \ \left\| \left(\tilde{I}_{2n}^{(s)'}(s) \right) \right\| \ell_2 \\[1ex] \left\| \left(\tilde{V}_{3n}^{(s)'}(s) \right) \right\| \ell_3 + \left\| \left(\tilde{Z}_{c_{n,m}}(s) \right)_{3,3} \right\| \ \left\| \left(\tilde{I}_{3n}^{(s)'}(s) \right) \right\| \ell_3 \\[1ex] \left\| \left(\tilde{V}_{3n}^{(s)'}(s) \right) \right\| \ell_3 + \left\| \left(\tilde{Z}_{c_{n,m}}(s) \right)_{3,3} \right\| \ \left\| \left(\tilde{I}_{3n}^{(s)'}(s) \right) \right\| \ell_3 \end{pmatrix} \right\| \right\|$$

(233)

c. Sources are rather uniform, that is, the variation of per-unit-length sources along the tubes is small. In this case, Equation 231 reduces to

$$\|((\tilde{V}_n^{(s)}(s))_u)\|$$

$$\leq \left\| \begin{pmatrix} [\|(\tilde{V}_{1n}^{(s)'}(z,s))\|_{\ell_1} + \|(\tilde{Z}_{c_{n,m}}(s))_{1,1}\| \|(\tilde{I}_{1n}^{(s)'}(z,s))\|_{\ell_1}]_{max} \\ [\|(\tilde{V}_{1n}^{(s)'}(z,s))\|_{\ell_1} + \|(\tilde{Z}_{c_{n,m}}(s))_{1,1}\| \|(\tilde{I}_{1n}^{(s)'}(z,s))\|_{\ell_1}]_{max} \\ [\|(\tilde{V}_{2n}^{(s)'}(z,s))\|_{\ell_2} + \|(\tilde{Z}_{c_{n,m}}(s))_{2,2}\| \|(\tilde{I}_{2n}^{(s)'}(z,s))\|_{\ell_2}]_{max} \\ [\|(\tilde{V}_{2n}^{(s)'}(z,s))\|_{\ell_2} + \|(\tilde{Z}_{c_{n,m}}(s))_{2,2}\| \|(\tilde{I}_{2n}^{(s)'}(z,s))\|_{\ell_2}]_{max} \\ [\|(\tilde{V}_{3n}^{(s)'}(z,s))\|_{\ell_3} + \|(\tilde{Z}_{c_{n,m}}(s))_{3,3}\| \|(\tilde{I}_{3n}^{(s)'}(z,s))\|_{\ell_3}]_{max} \\ [\|(\tilde{V}_{3n}^{(s)'}(z,s))\|_{\ell_3} + \|(\tilde{Z}_{c_{n,m}}(s))_{3,3}\| \|(\tilde{I}_{3n}^{(s)'}(z,s))\|_{\ell_3}]_{max} \end{pmatrix} \right\|$$

(234)

Thus, an upper bound for the source supervector can be calculated from Equations 230 through 234 in terms of the norms of per-unit-length voltage, current source vectors on the various tubes, and the characteristic-impedance matrices of the various tubes.

6. BOUNDS FOR COMBINED VOLTAGES, VOLTAGES, AND CURRENTS

Sections V.3 and V.4 established that the norms of the scattering and propagation supermatrices are less than or equal to one. Following the procedure used in Section IV.6 for the derivation of upper and lower bounds for combined voltages, voltages, and currents, similar relations for the present network can be written.

The upper and lower bounds for the combined voltages, voltages, and currents for a multiconductor transmission line with a branch (Fig. 4) are given by the following relations (for $s = j\omega$):

a. **Combined voltage vector for waves leaving junctions**

$$\|((\tilde{V}_n(0,s))_u)\|_\infty \leq \frac{\sqrt{N_s}\,\|((\tilde{S}_{n,m}(s))_{u,v})\|_2\,\|((\tilde{v}_n^{(s)}(s))_u)\|_\infty}{1 - \|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2\,\|((\tilde{S}_{n,m}(s))_{u,v})\|_2} \tag{235}$$

$$\|((\tilde{V}_n(0,s))_u)\|_\infty \geq \frac{\|((\tilde{v}_n^{(s)}(s))_u)\|_\infty}{\sqrt{N_s}[1 + \|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2\,\|((\tilde{S}_{n,m}(s))_{u,v})\|_2]\,\|((\tilde{S}_{n,m}(s))_{u,v})^{-1}\|_2} \tag{236}$$

b. **Combined voltage vector for waves entering junctions**

$$\|((\tilde{V}_n(L_u,s))_u)\|_\infty \leq \frac{\sqrt{N_s}\,\|((\tilde{V}_n(s))_u)\|_\infty}{1 - \|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2\,\|((\tilde{S}_{n,m}(s))_{u,v})\|_2} \tag{237}$$

$$\|((\tilde{V}_n(L_u,s))_u)\|_\infty \geq \frac{\|((\tilde{v}_n^{(s)}(s))_u)\|_\infty}{\sqrt{N_s}[1 + \|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2\,\|((\tilde{S}_{n,m}(s))_{u,v})\|_2]} \tag{238}$$

c. **Voltage vector at junctions**

$$\|((\tilde{v}_n^{(0)}(s))_u)\|_\infty \leq \frac{1}{2}\frac{\sqrt{N_s}[1 + \|((\tilde{S}_{n,m}(s))_{u,v})\|_2]\,\|((\tilde{v}_n^{(s)}(s))_u)\|_\infty}{1 - \|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2\,\|((\tilde{S}_{n,m}(s))_{u,v})\|_2} \tag{239}$$

$$\|((\tilde{V}_n^{(0)}(s))_u)\|_\infty \geq \frac{1}{2} \frac{\|((\tilde{V}_n^{(s)}(s))_u)\|_\infty [1 - \|((\tilde{S}_{n,m}(s))_{u,v})\|_2]}{\sqrt{N_s}[1 + \|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2 \ \|((\check{S}_{n,m}(s))_{u,v}\|_2]}$$

(240)

d. <u>Current vector at the junctions</u>

$$\|((\tilde{I}_n^{(0)}(s))_u)\|_\infty \leq \frac{1}{2} \frac{\sqrt{N_s} \ \|((\tilde{Y}_{c_{n,m}}(s))_{u,v})\|_2 [1 + \|((\tilde{S}_{n,m}(s))_{u,v})\|_2] \|((\tilde{V}_n^{(s)}(s))_u)\|_\infty}{1 - \|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2 \|((\tilde{S}_{n,m}(s))_{u,v})\|_2}$$

(241)

$$\|((\tilde{I}_n^{(0)}(s))_u)\|_\infty \geq \frac{1}{2} \frac{\|((\tilde{V}_n^{(s)}(s))_u)\|_\infty [1 - \|((\tilde{S}_{n,m}(s))_{u,v})\|_2]}{\sqrt{N_s}[1 + \|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2 \|((\tilde{S}_{n,m}(s))_{u,v})\|_2] \|((\check{Z}_{c_{n,m}}(s))_{u,v})\|_2}$$

(242)

87

VI. COMPUTATIONS OF SIGNAL BOUNDS FOR A UNIFORM MULTICONDUCTOR TRANSMISSION LINE

This section uses the results of previous sections, along with appendices, to compute upper and lower bounds for the voltage and the current at terminations of a uniform multiconductor transmission line. The examples in this section will help the reader understand fully the concepts and the procedures used in previous sections in computations of bounds, and also establish the tightness of these bounds.

1. PARAMETERS OF THE LINE

First consider a lossless line formed by three conductors plus a reference conductor (ground) as shown in Figure 5. The multiconductor transmission line consists of three identical conductors of 0.2 cm in diameter and 20 m in length above a perfectly conducting ground plane as shown in Figure 6.

The per-unit-length capacitance matrix of the line calculated from CAP CODE (Ref. 15) is

$$
(C_{n,m}) = \begin{pmatrix} 30.23 & -12.94 & -12.98 \\ -12.94 & 30.23 & -12.98 \\ -12.98 & -12.98 & 30.14 \end{pmatrix} \text{pF/m} \tag{243}
$$

The characteristic-impedance matrix of the line is given by

$$
(Z_{C_{n,m}}) = \frac{1}{v} (C_{n,m})^{-1} \tag{244}
$$

where v is the speed of propagation on the line. From Equations 243 and 244, the characteristic-impedance matrix of the line is
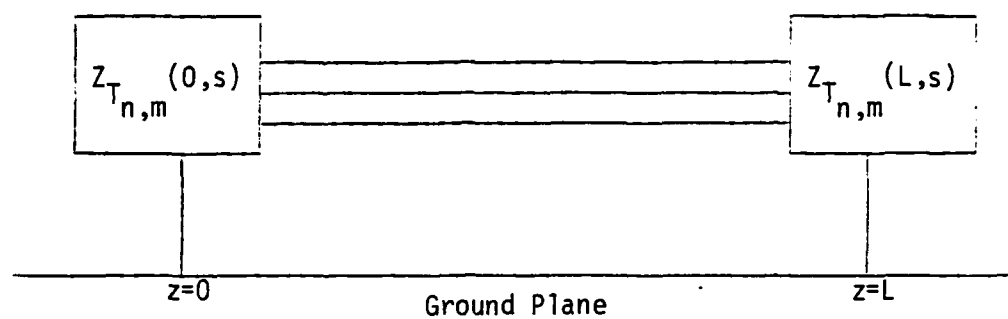
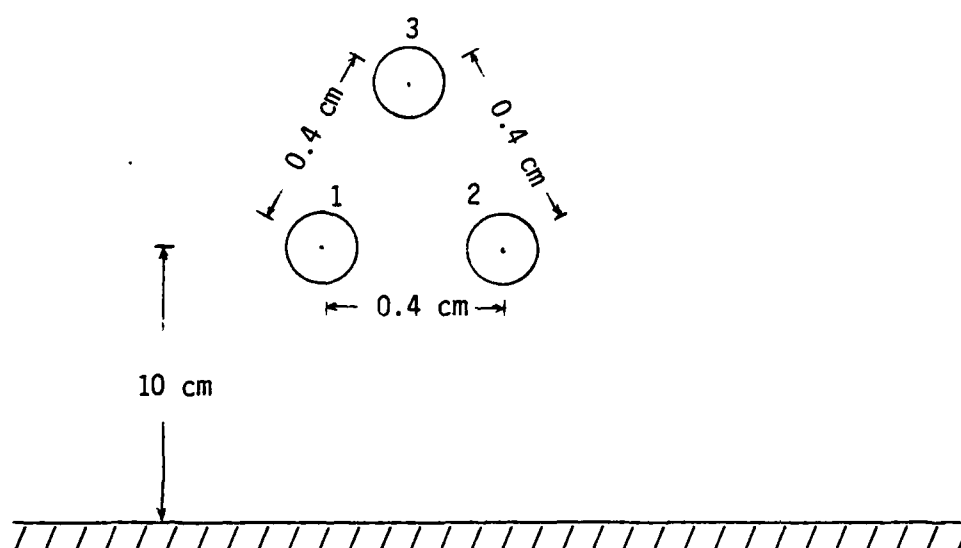Figure 5. A three-conductor line over a ground plane.

Figure 6. Cross section of a three-conductor line over a
ground plane.

$$\left(\tilde{Z}_{c_{n,m}}\right) = \begin{bmatrix} 311.377 & 234.163 & 234.941 \\ 234.163 & 311.377 & 234.941 \\ 234.941 & 234.941 & 312.953 \end{bmatrix} \text{ ohm} \qquad (245)$$

From Equation 245, the characteristic-admittance matrix of the line is

$$\left(\tilde{Y}_{c_{n,m}}\right) = \begin{pmatrix} 0.9069 & -0.3882 & -0.3894 \\ -0.3882 & 0.9069 & -0.3894 \\ -0.3894 & -0.3894 & 0.9042 \end{pmatrix} \times 10^{-2} \text{ mho} \qquad (246)$$

2.   BOUNDING RELATIONSHIPS

The four basic equations which determine the upper and lower bounds for voltages and currents at both terminations of the multiconductor line, as derived in Section IV (Eqs. 169, 170, 174, 175), are

$$\left\|\left(\left(\tilde{V}_n^{(0)}(s)\right)_u\right)\right\|_\infty \le \frac{1}{2} \frac{\sqrt{N_s}\left[1 + \left\|\left(\left(\tilde{S}_{n,m}(s)\right)_{u,v}\right)\right\|_2\right]\left\|\left(\left(\tilde{V}_n^{(s)}(s)\right)_u\right)\right\|_\infty}{1 - \left\|\left(\left(\tilde{\Gamma}_{n,m}(s)\right)_{u,v}\right)\right\|_2 \left\|\left(\left(\tilde{S}_{n,m}(s)\right)_{u,v}\right)\right\|_2} \qquad (247)$$

$$\left\|\left(\left(\tilde{I}_n^{(0)}(s)\right)_u\right)\right\|_\infty \le \frac{1}{2} \frac{\sqrt{N_s}\left\|\left(\left(\tilde{Y}_{c_{n,m}}(s)\right)_{u,v}\right)\right\|_2\left[1 + \left\|\left(\left(\tilde{S}_{n,m}(s)\right)_{u,v}\right)\right\|_2\right]\left\|\left(\left(\tilde{V}_n^{(s)}(s)\right)_u\right)\right\|_\infty}{1 - \left\|\left(\left(\tilde{\Gamma}_{n,m}(s)\right)_{u,v}\right)\right\|_2 \left\|\left(\left(\tilde{S}_{n,m}(s)\right)_{u,v}\right)\right\|_2}$$

$$\qquad (248)$$

$$\left\|\left(\left(\tilde{V}_n^{(0)}(s)\right)_u\right)\right\|_\infty \ge \frac{1}{2} \frac{\left[1 - \left\|\left(\left(\tilde{S}_{n,m}(s)\right)_{u,v}\right)\right\|_2\right]\left\|\left(\left(\tilde{V}_n^{(s)}(s)\right)_u\right)\right\|_\infty}{\sqrt{N_s}\left[1 + \left\|\left(\left(\tilde{\Gamma}_{n,m}(s)\right)_{u,v}\right)\right\|_2 \left\|\left(\left(\tilde{S}_{n,m}(s)\right)_{u,v}\right)\right\|_2\right]} \qquad (249)$$

$$\|((\tilde{I}_n^{(0)}(s))_u)\|_\infty \geq \frac{1}{2} \frac{[1 - \|((\tilde{S}_{n,m}(s))_{u,v})\|_2]\|((\tilde{v}_n^{(s)}(s))_u)\|_\infty}{\sqrt{N_s}[1 + \|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2 \ \|((\tilde{S}_{n,m}(s))_{u,v})\|_2] \ \|((\tilde{Z}_{c_{n,m}}(s))_{u,v})\|_2}$$

(250)

For a lossless line, from Equation 115, the 2 norm of the propagation supermatrix is equal to unity, i.e.,

$$\|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2 = 1$$

(251)

Substitution of Equation 251 into Equations 247 through 250 yields

$$\|((\tilde{v}_n^{(0)}(s))_u)\|_\infty \leq \frac{1}{2} \frac{\sqrt{N_s}[1 + \|((\tilde{S}_{n,m}(s))_{u,v})\|_2] \ \|((\tilde{v}_n^{(s)}(s))_u)\|_\infty}{1 - \|((\tilde{S}_n(s))_{u,v})\|_2}$$

(252)

$$\|((\tilde{I}_n^{(0)}(s))_u)\|_\infty \leq \frac{1}{2} \frac{\sqrt{N_s} \ \|((\tilde{Y}_{c_{n,m}}(s))_{u,v})\|_2 [1 + \|((\tilde{S}_{n,m}(s))_{u,v})\|_2]\|((\tilde{v}_n^{(s)}(s))_u)\|_\infty}{1 - \|((\tilde{S}_{n,m}(s))_{u,v})\|_2}$$

(253)

$$\|((\tilde{v}_n^{(0)}(s))_u)\|_\infty \geq \frac{1}{2} \frac{[1 - \|((\tilde{S}_{n,m}(s))_{u,v})\|_2] \ \|((\tilde{v}_n^{(s)}(s))_u)\|_\infty}{\sqrt{N_s}[1 + \|((\tilde{S}_{n,m}(s))_{u,v})\|_2]}$$

(254)

$$\|((\tilde{I}_n^{(0)}(s))_u)\|_\infty \geq \frac{1}{2} \frac{[1 - \|((\tilde{S}_{n,m}(s))_{u,v})\|_2] \ \|((\tilde{v}_n^{(s)}(s))_u)\|_\infty}{\sqrt{N_s}[1 + \|((\tilde{S}_{n,m}(s))_{u,v})\|_2] \ \|((\tilde{Y}_{c_{n,m}}(s))_{u,v})\|_2}$$

(255)

Note that the expressions for upper and lower bounds in Equations 252 through 255 contain two factors; the first factor depends only on the characteristic properties of the multiconductor transmission line and the load impedances, and

91

the second factor depends on the distributed sources on the line. For a loss-less line with resistive loads the first factor is independent of the frequency. Later in this section are shown the ways to optimize the first factor in order to improve upon the tightness of bounds.

3.   COMPUTATIONS OF SIGNAL LEVELS AND BOUNDS

The upper and lower bounds for the multiconductor line in Figure 5 can be calculated for various load and source configurations. Let us consider the configuration shown in Figure 7 where the line is terminated at both ends in diagonal, resistive loads and is excited by three lumped series voltage sources at z = 10 m (midway between termination). We will consider the following load configurations.

Example 1. Let all the termination resistors be 50 Ω at both ends the line and all three lumped voltage sources be equal to constant 1 V each, so that

$$(\tilde{Z}_{T_{n,m}}(0,s)) = (\tilde{Z}_{T_{n,m}}(L,s)) = \begin{pmatrix} 50.0 & 0.0 & 0.0 \\ 0.0 & 50.0 & 0.0 \\ 0.0 & 0.0 & 50.0 \end{pmatrix} \qquad (256)$$

and

$$((\tilde{V}_n^{(s)}(s))_u) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \qquad (257)$$

The voltages and currents at the loads are calculated using QV7TA code (Ref. 5). The voltages and currents at the terminations are shown in Figures 8 through 19 as a function of frequency. From these responses, note that the maximum voltage and current are
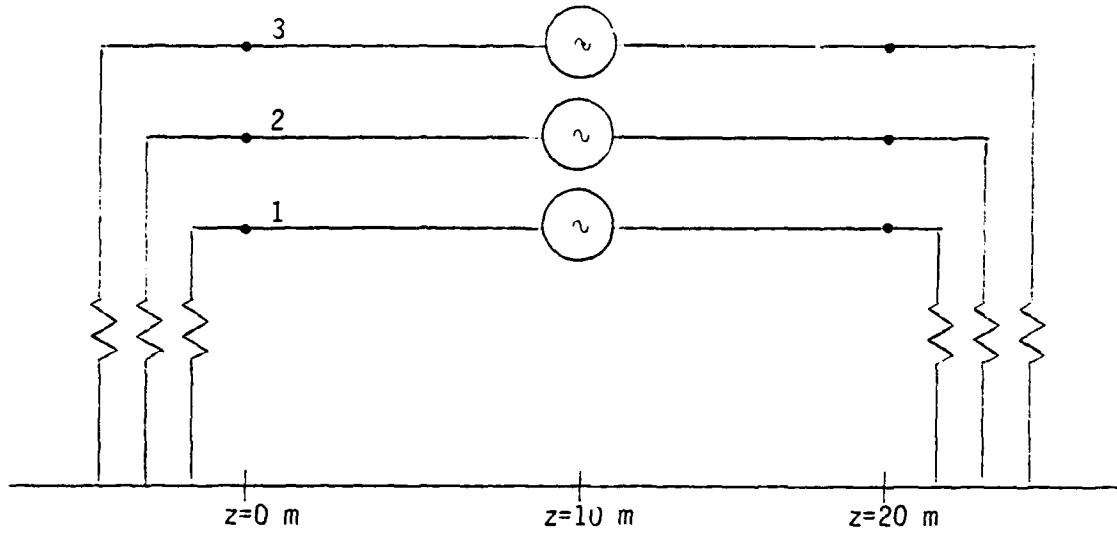
92

Figure 7. A three-conductor line with diagonal, resistive loads.

$$\left| \left(\tilde{V}_n^{(0)}(s) \right)_u \right|_{max} = 0.5 \text{ V}$$

$$\left| \left(\tilde{I}_n^{(0)}(s) \right)_u \right|_{max} = 0.01 \text{ A} \tag{258}$$

Using Equations 252 through 255, compute the upper and lower bounds for voltage and current, and compare these bounds with the calculated results.

The scattering matrices at the junctions (terminations) can be calculated from Equations 23 and 24 and are given by

$$\left(\tilde{S}_{n,m}(s) \right)_{1,2} = \left(\tilde{S}_{n,m}(s) \right)_{2,1} = \begin{pmatrix} 0.435 & 0.221 & 0.222 \\ 0.221 & 0.435 & 0.222 \\ 0.222 & 0.222 & 0.437 \end{pmatrix} \tag{259}$$

Since the matrix in Equation 259 is real symmetric, its 2 norm is equal to its maximum eigenvalue, i.e.,

$$\left\| \left(\tilde{S}_{n,m}(s) \right)_{1,2} \right\|_2 = \left\| \left(\tilde{S}_{n,m}(s) \right)_{2,1} \right\|_2 = 0.879 \tag{260}$$

93
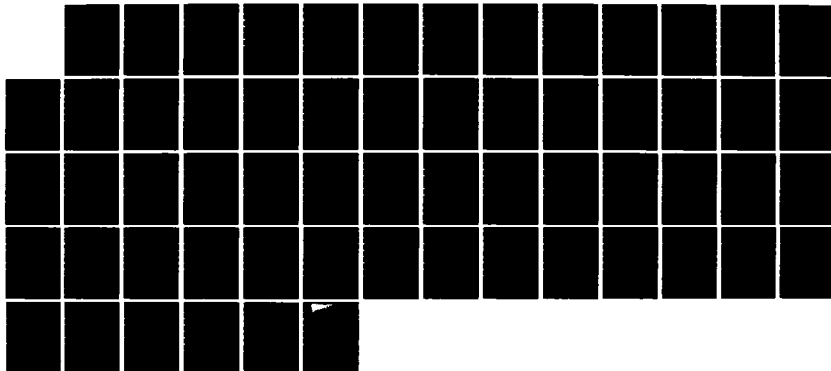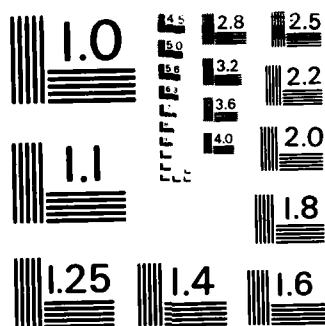
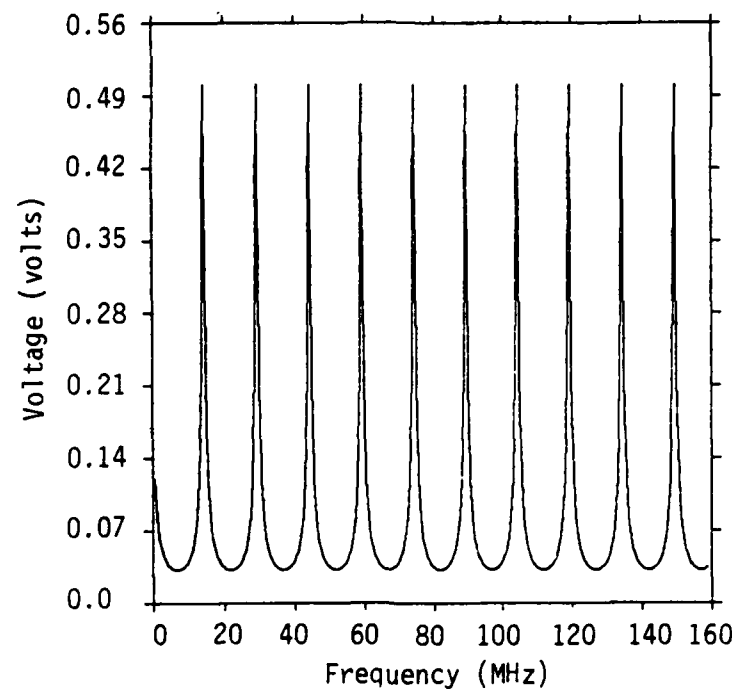MICROCOPY RESOLUTION TEST CHART

NATIONAL BUREAU OF STANDARDS-1963-A

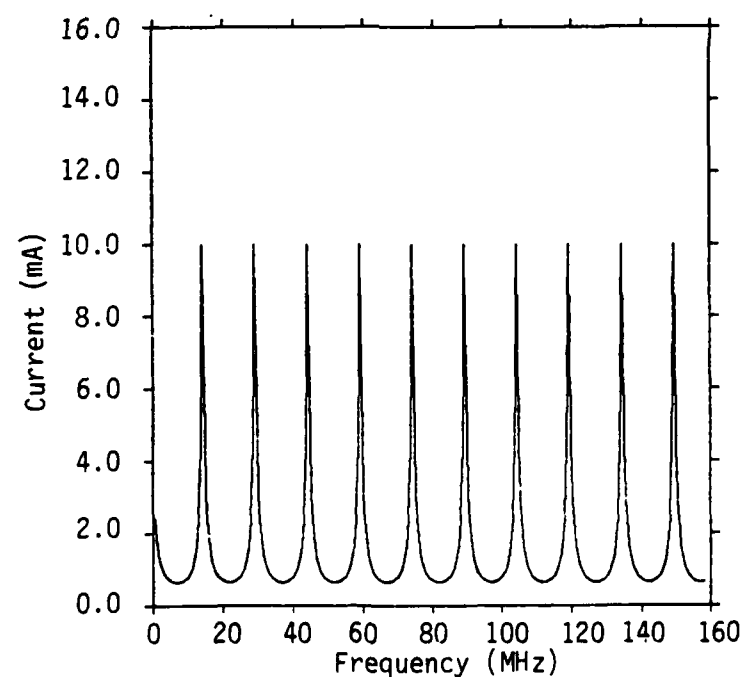Figure 8. The voltage waveform on wire 1 at z = 0.
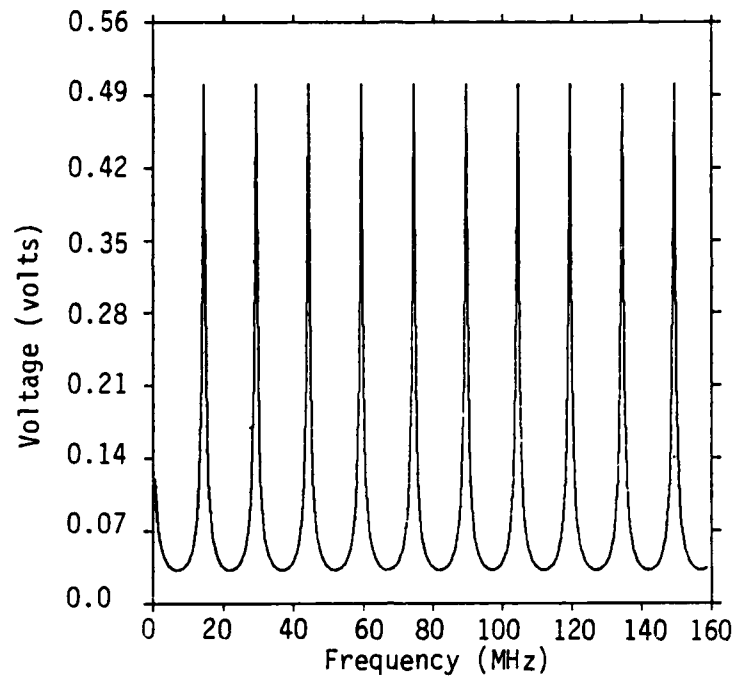

Figure 9. The current waveform on wire 1 at z = 0.

94

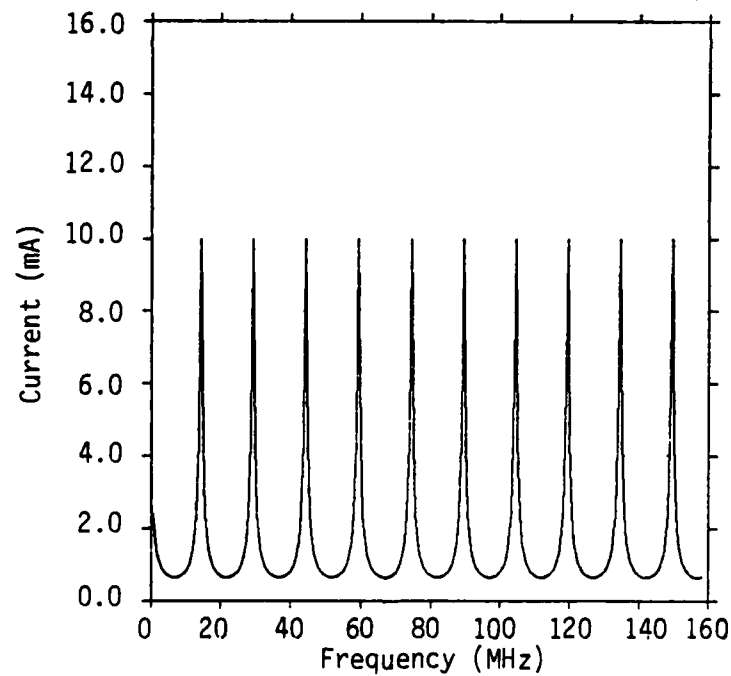Figure 10.   The voltage waveform on wire 2 at z = 0.

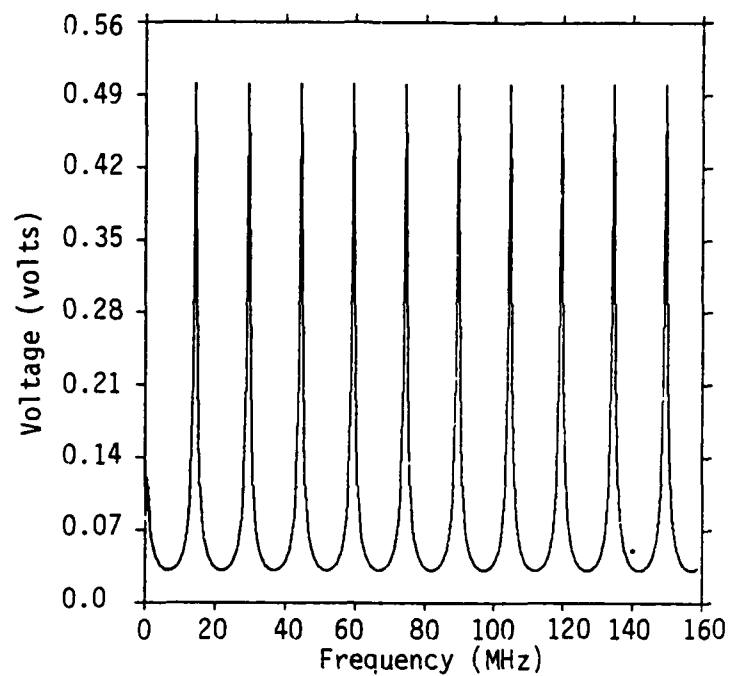
Figure 11.   The current waveform on wire 2 at z = 0.

95

Figure 12. The voltage waveform on wire 3 at z = 0.

Figure 13. The current waveform on wire 3 at z = 0.

Figure 14.   The voltage waveform on wire 1 at z = 20 m.

Figure 15.   The current waveform on wire 1 at z = 20 m.

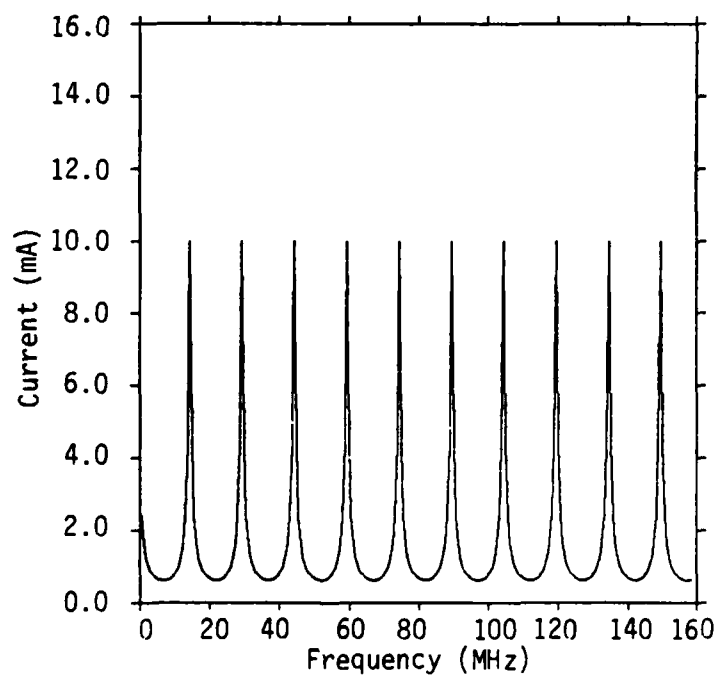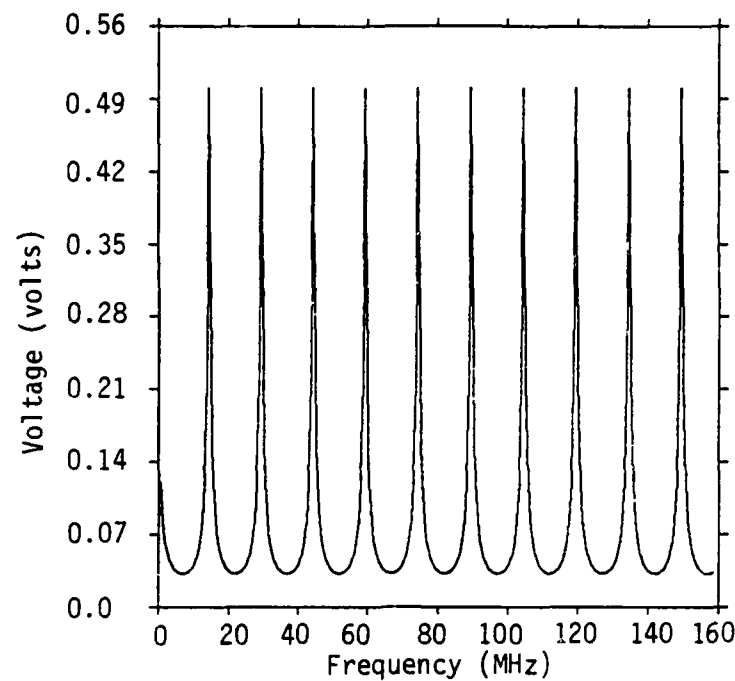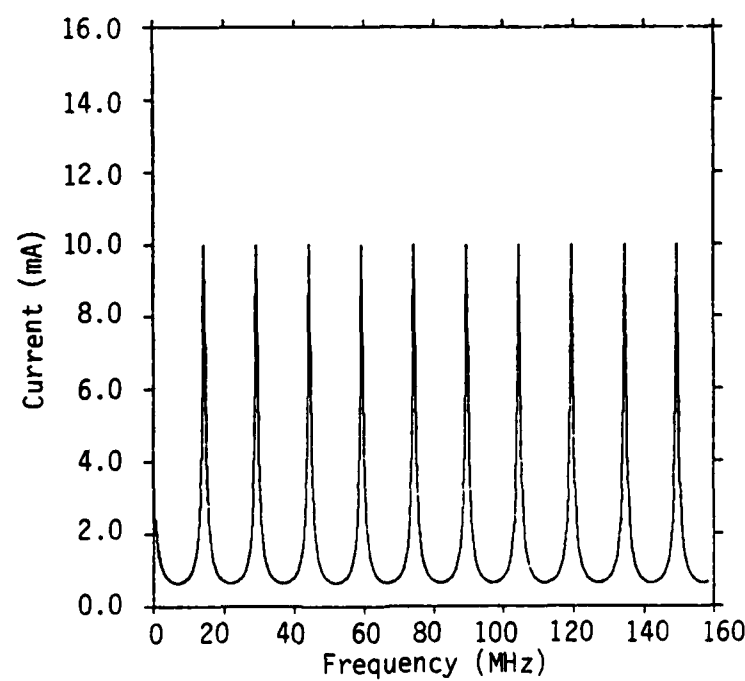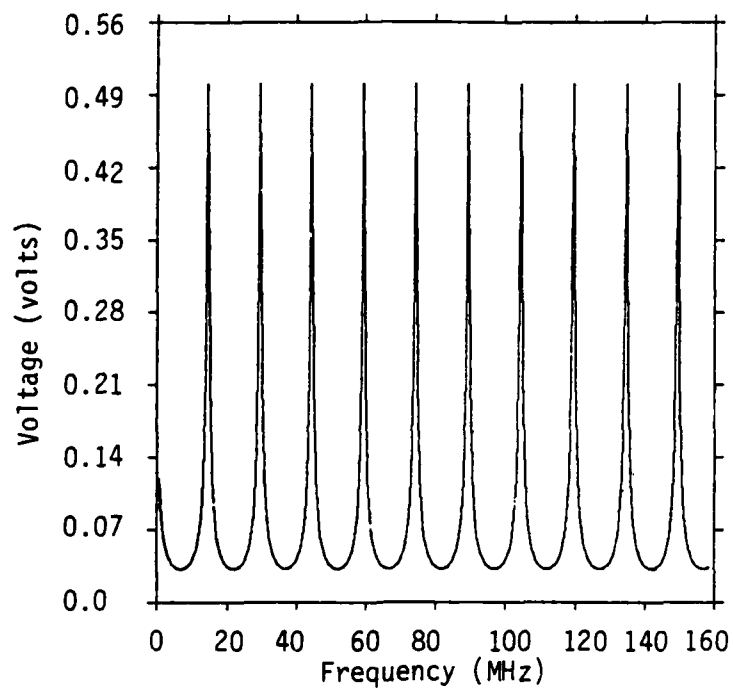Figure 16.   The voltage waveform on wire 2 at z = 20 m.
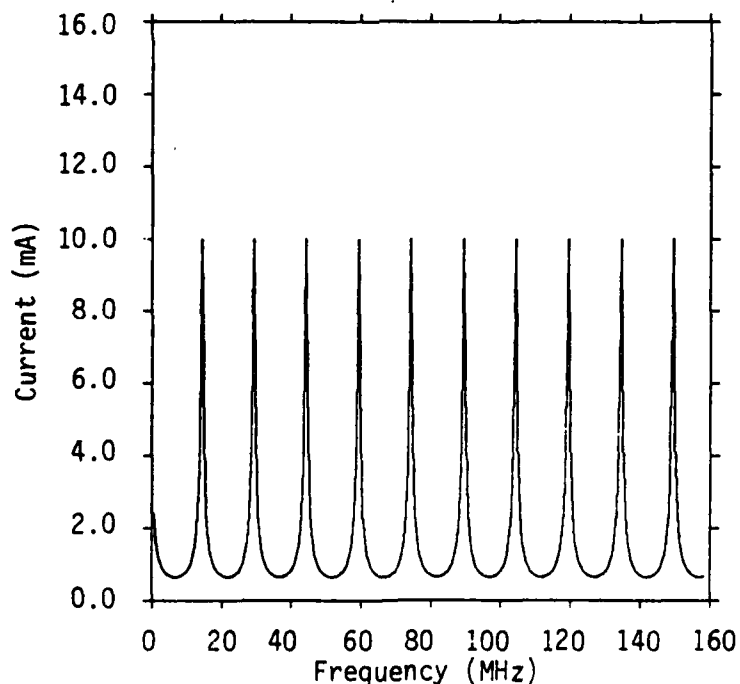

Figure 17.   The current waveform on wire 2 at z = 20 m.
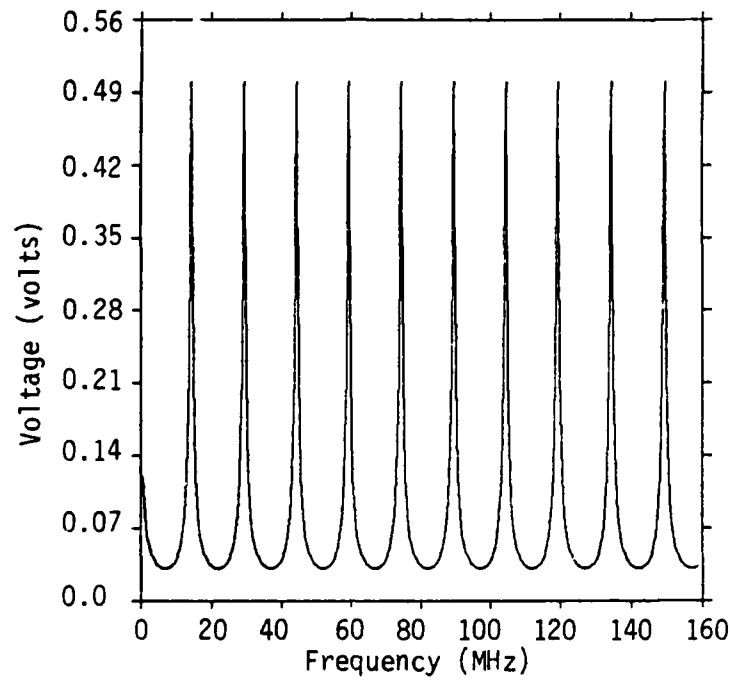
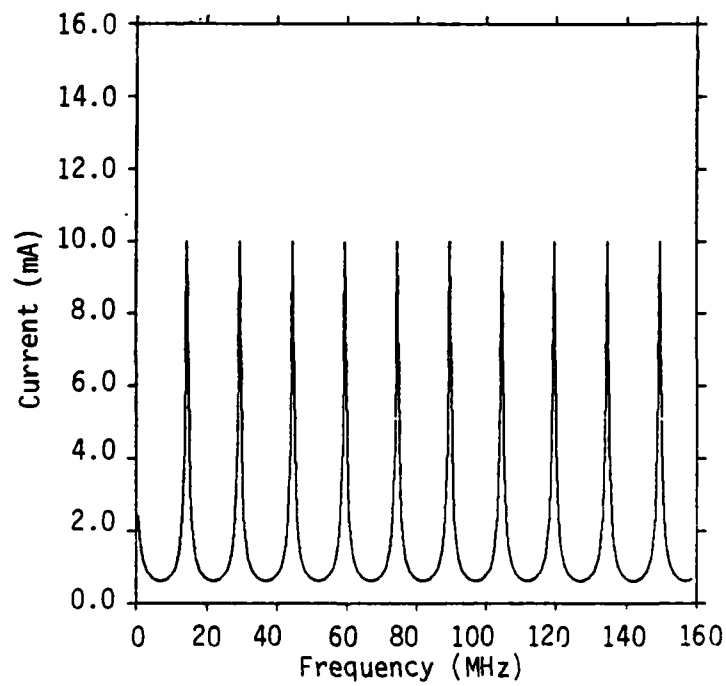98

Figure 18.   The voltage waveform on wire 3 at z = 20 m.

Figure 19.   The current waveform on wire 3 at z = 20 m.

99

From Equations 107 and 260, one has

$$\|((\tilde{S}_{n,m}(s))_{u,v})\|_2 = 0.879 \tag{261}$$

From Equation A60, the ∞ norm of the source supervector is given by

$$\|((\tilde{v}_n^{(s)}(s))_u)\|_\infty = 1 \tag{262}$$

The eigenvalues of the characteristic-admittance matrix are 0.01295, 0.01295, and 0.00128; and from Equations 177 and A70, the 2 norm of the characteristic-admittance supermatrix is obtained as

$$\|((\tilde{Y}_{c_{n,m}}(s))_{u,v})\|_2 = 0.01295 \tag{263}$$

Similarly, one obtains the 2 norm of the characteristic-impedance supermatrix as

$$\|((\tilde{Z}_{c_{n,m}}(s))_{u,v})\|_2 = 781.27 \tag{264}$$

Substituting Equations 261 through 264 into Equations 252 through 255, one obtains the upper and lower bounds for voltage and current at terminations as

$$\|((\tilde{v}_n^{(0)}(s))_u)\|_\infty \leq 19.0189 \text{ V}$$

$$\|((\tilde{I}_n^{(0)}(s))_u)\|_\infty \leq 0.2463 \text{ A}$$

$$\|((\tilde{v}_n^{(0)}(s))_u)\|_\infty \geq 0.01314 \text{ V} \tag{265}$$

$$\|((\tilde{I}_n^{(0)}(s))_u)\|_\infty \geq 0.17 \times 10^{-4} \text{ A}$$

As noted earlier in Section III, the upper and lower bounds for voltage and current in Equation 265 are the upper and lower bounds on the magnitude of the largest voltage and current at the terminations, respectively. The lower bounds should not be confused with the smallest voltage and current at the terminations.

First consider a few other terminations before studying the reasons for the bounds in Equation 265 to be loose compared to the calculated values.

Example 2. Consider that the multiconductor line in Figure 7 is terminated in resistive loads such that

$$
(\tilde{Z}_{T_{n,m}}(0,s)) = (\tilde{Z}_{T_{n,m}}(L,s)) = \begin{pmatrix} 200.0 & 0.0 & 0.0 \\ 0.0 & 200.0 & 0.0 \\ 0.0 & 0.0 & 200.0 \end{pmatrix} \quad (266)
$$

the scattering matrices at the terminations for this case are

$$
(\tilde{S}_{n,m}(s))_{1,2} = (\tilde{S}_{n,m}(s))_{2,1} = \begin{pmatrix} -0.0986 & 0.344 & 0.345 \\ 0.344 & -0.0986 & 0.345 \\ 0.345 & 0.345 & -0.0963 \end{pmatrix} \quad (267)
$$

From Equations 107 and 267, one has

$$
\|((\tilde{S}_{n,m}(s))_{u,v})\|_2 = 0.5915 \quad (268)
$$

The maximum voltage and current calculated from QV7TA code occur on wire 1, and the voltage and current responses on wire 1 are shown in Figures 20 and 21. From symmetry, the responses at z = 0 and z = 20 m are identical. From Figures 20 and 21, note that

101
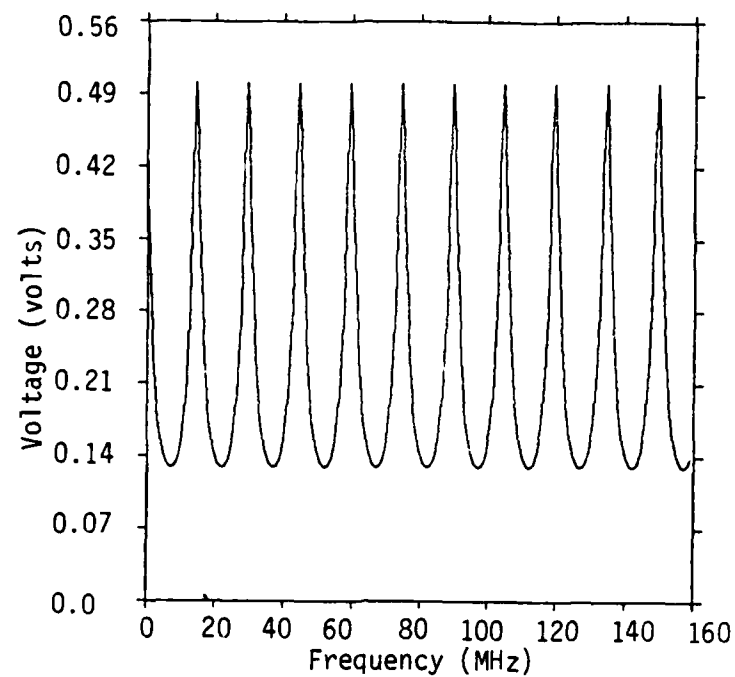
Figure 20.   The voltage waveform on wire 1 at z = 0.
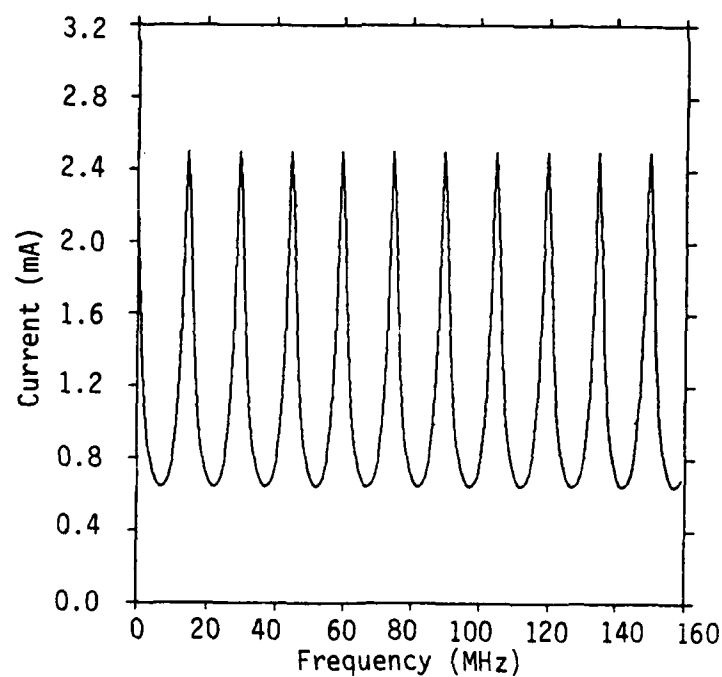
Figure 21.   The current waveform on wire 1 at z = 0.

102

$$| \; (\tilde{V}_n^{(0)}(s))_u \; |_{max} = 0.5 \; V$$

$$| \; (\tilde{I}_n^{(0)}(s))_u \; |_{max} = 2.5 \times 10^{-3} \; A$$

(269)

Substituting Equations 262, 263, 264, and 268 into Equations 252 through 255, one obtains

$$\|((\tilde{V}_n^{(0)}(s))_u)\|_\infty \leq 4.7715 \; V$$

$$\|((\tilde{I}_n^{(0)}(s))_u)\|_\infty \leq 0.0618 \; A$$

(270)

$$\|((\tilde{V}_n^{(0)}(s))_u)\|_\infty \geq 0.0524 \; V$$

$$\|((\tilde{I}_n^{(0)}(s))_u)\|_\infty \geq 6.7 \times 10^{-5} \; A$$

Example 3. Consider that the multiconductor line in Figure 7 is terminated in resistive loads such that

$$(\tilde{Z}_{T_{n,m}}(0,s)) = \begin{pmatrix} 50.0 & 0.0 & 0.0 \\ 0.0 & 50.0 & 0.0 \\ 0.0 & 0.0 & 50.0 \end{pmatrix}$$

(271)

$$(\tilde{Z}_{T_{n,m}}(L,s)) = \begin{pmatrix} 200.0 & 0.0 & 0.0 \\ 0.0 & 200.0 & 0.0 \\ 0.0 & 0.0 & 200.0 \end{pmatrix}$$

(272)

From Equations 259 and 267 one has

$$(\tilde{S}_{n,m}(s))_{1,2} = 0.879$$

(273)

$$(\tilde{S}_{n,m}(s))_{2,1} = 0.5915$$

(274)

From Equations 273, 274 and 107 one obtains

$$\|((\tilde{S}_{n,m}(s))_{u,v})\| = 0.879 \tag{275}$$

The voltage and current waveforms on wire 1 at $z = 0$ and $z = 20$ m are shown in Figures 22 and 23, and 24 and 25, respectively. Because of symmetry, the responses on wires 2 and 3 are the same as on wire 1. Figures 20 through 23 show that

$$| (\tilde{v}_n^{(0)}(s))_u |_{max} = 0.8 \text{ V}$$

$$| (\tilde{I}_n^{(0)}(s))_u |_{max} = 0.004 \text{ A} \tag{276}$$

Substituting Equations 262, 263, 264 and 275 into Equations 252 through 255 gives

$$\|((\tilde{v}_n^{(0)}(s))_u)\|_\infty \leq 19.0189 \text{ V}$$

$$\|((\tilde{I}_n^{(0)}(s))_u)\|_\infty \leq 0.2463 \text{ A}$$

$$\|((\tilde{v}_n^{(0)}(s))_u)\|_\infty \geq 0.01314 \text{ V} \tag{277}$$

$$\|((\tilde{I}_n^{(0)}(s))_u)\|_\infty \geq 0.17 \times 10^{-4} \text{ A}$$

Note that these bounds are the same as for example 1. The bounds depend mainly on the scattering matrices at the junctions, and the smaller values of the terminations determine the upper and lower bounds.

Example 4. As the last example, consider that the multiconductor line in Figure 7 is terminated in its characteristic impedance at both ends, so that

$$(\tilde{Z}_{T_{n,m}}(0,s)) = (\tilde{Z}_{T_{n,m}}(L,s)) = (\tilde{Z}_{c_{n,m}}(s)) \tag{278}$$

104
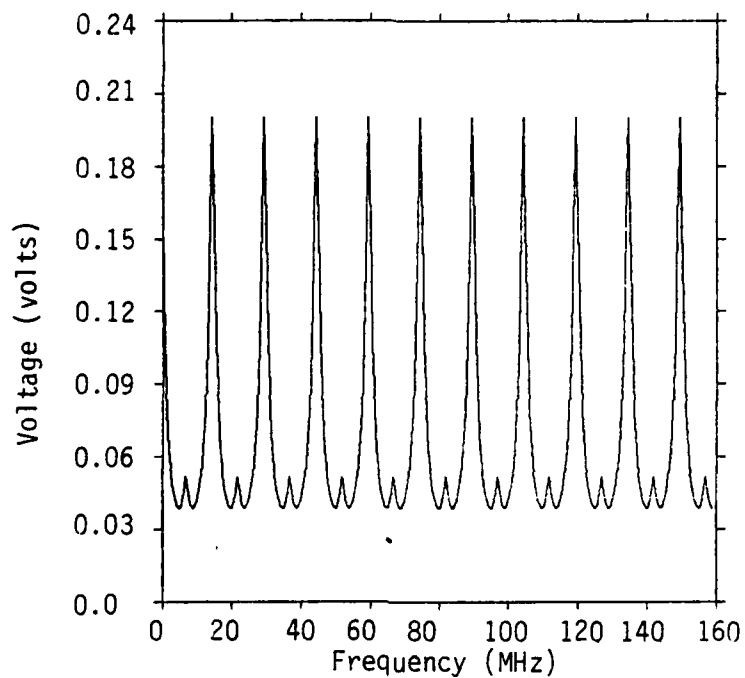
Figure 22. The voltage waveform on wire 1 at z = 0.
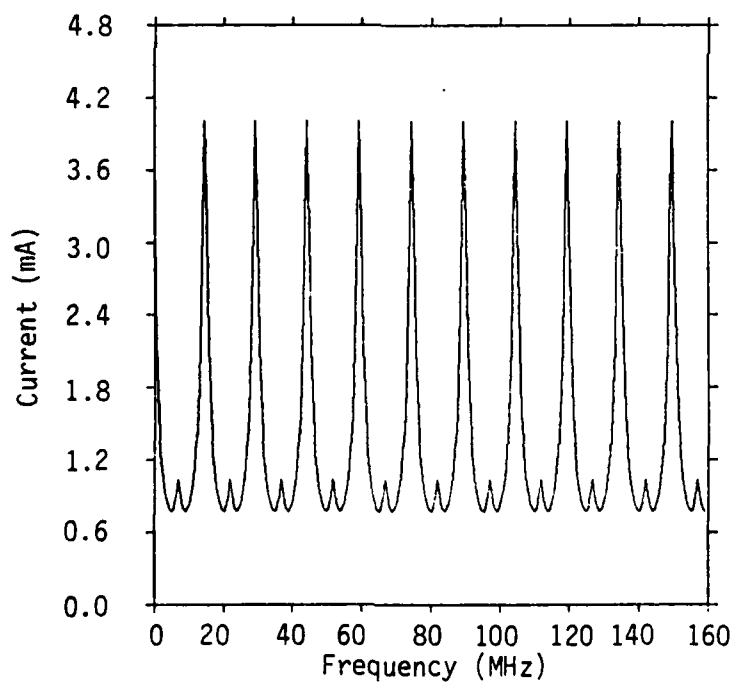
Figure 23. The current waveform on wire 1 at z = 0.
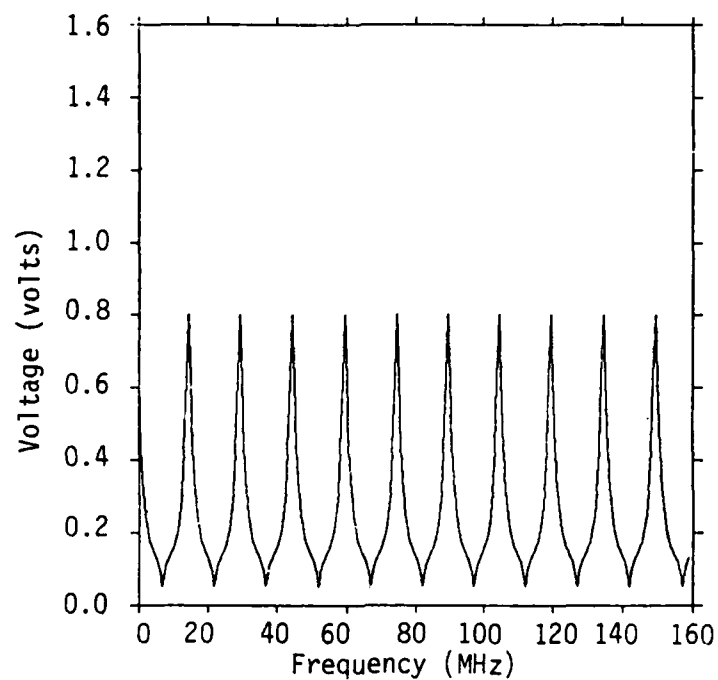
Figure 24. The voltage waveform on wire 1 at z = 20 m.
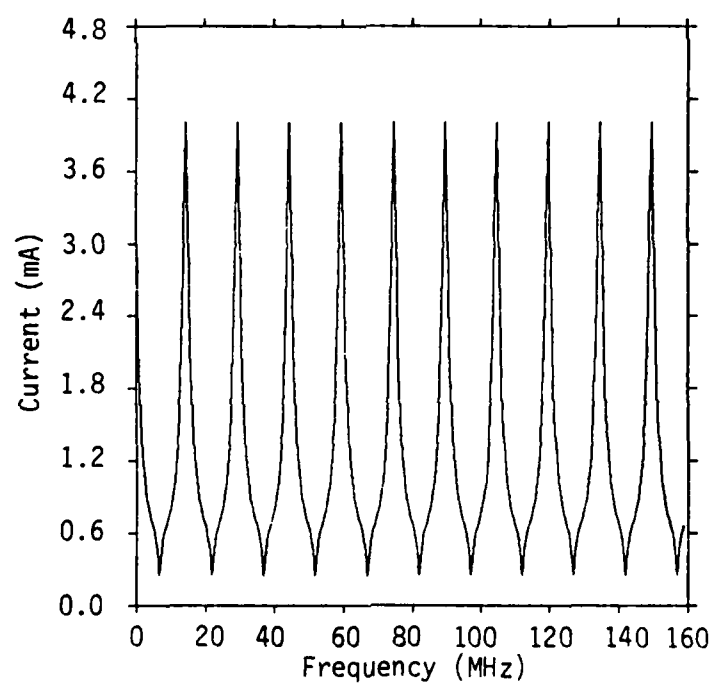
Figure 25. The voltage waveform on wire 1 at z = 20 m.

The maximum voltage and current occur on wire 1. The voltage and current waveforms on wire 1 at z = 0 are shown in Figures 26 and 27. These responses show that

$$| \ (\tilde{V}_n^{(0)}(s))_u \ |_{max} = 0.5 \ V$$

$$| \ (\tilde{I}_n^{(0)}(s))_u \ |_{max} = 0.6465 \times 10^{-3} \ A$$

(279)

The scattering matrices at the terminations for this case are null matrices, that is,

$$(\tilde{S}_{n,m}(s))_{1,2} = (\tilde{S}_{n,m}(s))_{2,1} = (0_{n,m})$$

(280)

Therefore

$$\|((\tilde{S}_{n,m}(s))_{u,v})\|_2 = 0$$

(281)

Substituting Equations 262, 263, 264 and 281 into Equations 252 through 255, one obtains

$$\|((\tilde{V}_n^{(0)}(s))_u)\|_\infty \leq 1.2247 \ V$$

$$\|((\tilde{I}_n^{(0)}(s))_u)\|_\infty \leq 0.01586 \ A$$

$$\|((\tilde{V}_n^{(0)}(s))_u)\|_\infty \geq 0.2041 \ V$$

$$\|((\tilde{I}_n^{(0)}(s))_u)\|_\infty \geq 0.26 \times 10^{-3} \ A$$

(282)
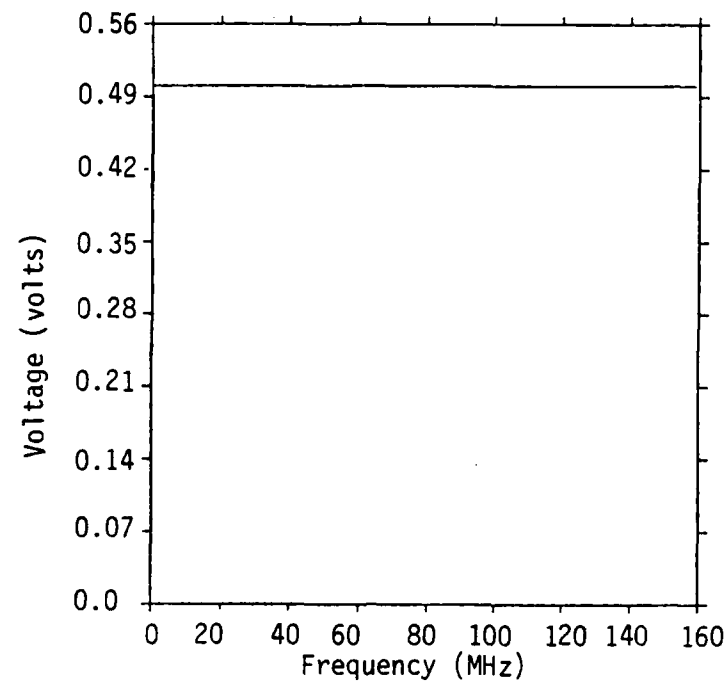
Figure 26.   The voltage waveform on wire 1 at z = 0.
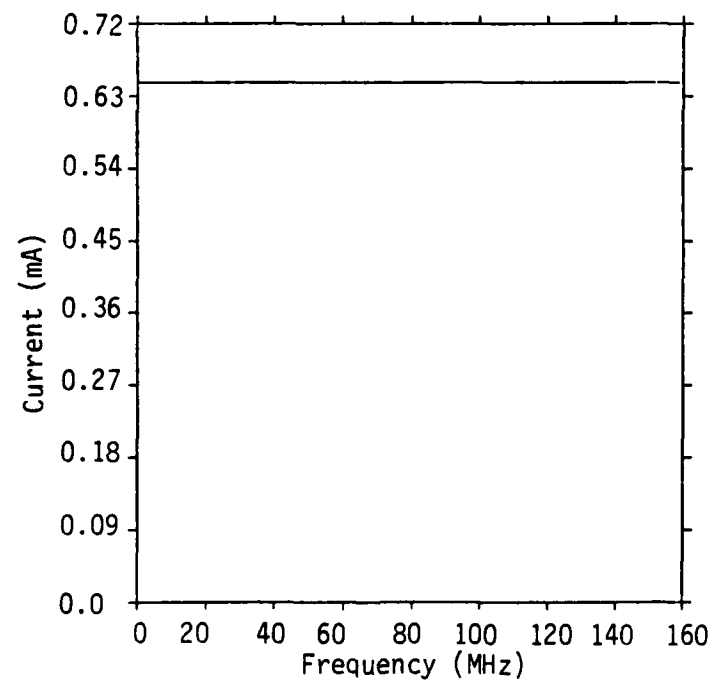
Figure 27.   The current waveform on wire 1 at z = 0.

108

4. ANALYSIS OF THE CALCULATED BOUNDS

In Equations 247 through 250, a factor $\sqrt{N_s}$ appears because the 2 norms of the scattering, propagation, and characteristic-impedance supermatrices are used instead of the ∞ norms. In all four examples, note that the ∞ norms of the scattering matrices are less than one and, for a lossless homogeneous case, the ∞ norm of the propagation supermatrix is equal to one. Therefore, the upper and lower bounds for voltages and currents can be expressed in terms of the ∞ norms of the other quantities. Following the same procedure used in the derivation of Equations 252 through 255 yields

$$\|((\tilde{V}_n^{(0)}(s))_u)\|_\infty \le \frac{1}{2} \frac{[1 + \|((\tilde{S}_{n,m}(s))_{u,v})\|_\infty]\|((\tilde{V}_n^{(s)}(s))_u)\|_\infty}{1 - \|((\tilde{S}_{n,m}(s))_{u,v})\|_\infty} \tag{283}$$

$$\|((\tilde{I}_n^{(s)}(s))_u)\|_\infty \le \frac{1}{2} \frac{\|((\tilde{Y}_{c_{n,m}}(s))_{u,v})\|_\infty [1 + \|((\tilde{S}_{n,m}(s))_{u,v})\|_\infty]\|((\tilde{V}_n^{(s)}(s))_u)\|_\infty}{1 - \|((\tilde{S}_{n,m}(s))_{u,v})\|_\infty} \tag{284}$$

$$\|((\tilde{V}_n^{(0)}(s))_u)\|_\infty \ge \frac{1}{2} \frac{[1 - \|((\tilde{S}_{n,m}(s))_{u,v})\|_\infty] \|((\tilde{V}_n^{(s)}(s))_u)\|_\infty}{1 + \|((\tilde{S}_{n,m}(s))_{u,v})\|_\infty} \tag{285}$$

$$\|((\tilde{I}_n^{(0)}(s))_u)\|_\infty \ge \frac{1}{2} \frac{[1 - \|((\tilde{S}_{n,m}(s))_{u,v})\|_\infty] \|((\tilde{V}_n^{(s)}(s))_u)\|_\infty}{[1 + \|((\tilde{S}_{n,m}(s))_{u,v})\|_\infty] \|((\tilde{Z}_{c_{n,m}}(s))_{u,v})\|_\infty} \tag{286}$$

From Equations 245, 246, and A10, one obtains

$$\|((\tilde{Y}_{c_{n,m}}(s))_{u,v})\|_\infty = 0.01684 \tag{287}$$

$$\|((\tilde{Z}_{c_{n,m}}(s))_{u,v})\|_\infty = 782.835 \tag{288}$$

The ∞ norms of the scattering supermatrices for the four examples considered earlier are:

a. Example 1--50-Ω termination

$$\|((\tilde{S}_{n,m}(s))_{u,v})\|_{\infty} = 0.88 \qquad (289)$$

b. Example 2--200-Ω termination

$$\|((\tilde{S}_{n,m}(s))_{u,v})\|_{\infty} = 0.7876 \qquad (290)$$

c. Example 3--50-Ω and 200-Ω termination

$$\|((\tilde{S}_{n,m}(s))_{u,v})\|_{\infty} = 0.88 \qquad (291)$$

d. Example 4--characteristic-impedance termination

$$\|((\tilde{S}_{n,m}(s))_{u,v})\|_{\infty} = 0 \qquad (292)$$

Substituting Equations 287 through 292 into Equations 283 through 286 upper and lower bounds for voltage and current for the four examples are obtained as:

a. Example 1--50-Ω termination

$$\|((\tilde{V}_n^{(0)}(s))_u)\|_{\infty} \leq 7.833 \text{ V}$$

$$\|((\tilde{I}_n^{(0)}(s))_u)\|_{\infty} \leq 0.1319 \text{ A}$$

$$\|((\tilde{V}_n^{(0)}(s))_u)\|_{\infty} \geq 0.0319 \text{ V} \qquad (293)$$

$$\|((\tilde{I}_n^{(0)}(s))_u)\|_{\infty} \geq 0.41 \times 10^{-4} \text{ A}$$

b. Example 2--200-Ω termination

$$\|((\tilde{V}_n^{(0)}(s))_u)\|_\infty \leq 4.208 \text{ V}$$

$$\|((\tilde{I}_n^{(0)}(s))_u)\|_\infty \leq 0.0709 \text{ A}$$

$$\|((\tilde{V}_n^{(0)}(s))_u)\|_\infty \geq 0.0594 \text{ V}$$

$$\|((\tilde{I}_n^{(0)}(s))_u)\|_\infty \geq 0.76 \times 10^{-4} \text{ A}$$

(294)

c. Example 3--50-Ω and 200-Ω terminations

$$\|((\tilde{V}_n^{(0)}(s))_u)\|_\infty \leq 7.833 \text{ V}$$

$$\|((\tilde{I}_n^{(0)}(s))_u)\|_\infty \leq 0.1319 \text{ A}$$

$$\|((\tilde{V}_n^{(0)}(s))_u)\|_\infty \geq 0.0319 \text{ V}$$

$$\|((\tilde{I}_n^{(0)}(s))_u)\|_\infty \geq 0.41 \times 10^{-4} \text{ A}$$

(295)

d. Example 4--characteristic-impedance termination

$$\|((\tilde{V}_n^{(0)}(s))_u)\|_\infty \leq 0.5 \text{ V}$$

$$\|((\tilde{I}_n^{(0)}(s))_u)\|_\infty \leq 0.0084 \text{ A}$$

$$\|((\tilde{V}_n^{(0)}(s))_u)\|_\infty \geq 0.5 \text{ V}$$

$$\|((\tilde{I}_n^{(0)}(s))_u)\|_\infty \geq 0.639 \times 10^{-3} \text{ A}$$

(296)

Comparison of Equations 265 and 293 shows that there is an improvement in the bounds by a factor of approximately $\sqrt{6}$. This is due to the fact that the ∞ and 2 norms of the scattering, characteristic-impedance, and admittance supermatrices

111

are approximately the same, and when ∞ norms of supermatrices are used in Equations 283 through 286, the constant multiplier $\sqrt{N_s}$ drops out. But in the case of example 2, this is not the case. A comparison between Equations 270 and 294 shows that there is no significant improvement in bounds when ∞ norms are used instead of 2 norms. For examples 3 and 4, the bounds are improved by a factor of $\sqrt{6}$, as is evident from comparisons between Equations 277 and 295, and 282 and 296, respectively.

Note that the upper and lower bounds on voltage for example 4 (characteristic-impedance termination) are exactly equal to the calculated values. A comparison of bounds with calculated values in examples 1 through 4 indicates that tighter bounds are obtained when the termination impedances are closer to the characteristic-impedance matrix of the multiconductor line. This is because a factor $[1 + \|((\tilde{S}_{n,m}(s))_{u,v})\|_2]/[1 - \|((\tilde{S}_{n,m}(s))_{u,v})\|_2]$ appears in expressions for upper bounds on voltage and current and a factor $[1 - \|((\tilde{S}_{n,m}(s))_{u,v})\|_2]/[1 + \|((\tilde{S}_{n,m}(s))_{u,v})\|_2]$ appears in expressions for lower bounds on voltage and current. The higher values of the 2 norm of the scattering supermatrix result in higher upper bounds and smaller lower bounds.

Another source of error in the calculation of upper and lower bounds for voltage and current is the use of inequality in Equation 132 in deriving the Equations 247 through 250. For the four examples considered in this section, the left-hand side of Equation 132 will be calculated and compared with the bounds obtained by using the left-hand side of the equation rather than the right-hand side. The modified upper and lower bounds for voltage and current can be written as

$$\|((\tilde{v}_n^{(0)}(s))_u)\|_\infty \leq \frac{1}{2}\sqrt{N_s} \, \|[((\tilde{S}_{n,m}(s))_{u,v}) + ((P_{n,m})_{u,v})]\|_2$$

$$\cdot \|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]^{-1}\|_2 \|((\tilde{v}_n^{(s)}(s))_u)\|_\infty$$

(297)

$$\|((\tilde{I}_n^{(0)}(s))_u)\|_\infty \leq \frac{1}{2}\sqrt{N_s} \, \|((\tilde{Y}_{c_{n,m}}(s))_{u,v})\|_2 \|[((\tilde{S}_{n,m}(s))_{u,v}) - ((P_{n,m})_{u,v})]\|_2$$

$$\cdot \|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]^{-1}\|_2 \|((\tilde{v}_n^{(s)}(s))_u)\|_\infty$$

(298)

$$\|((\tilde{v}_n^{(0)}(s))_u)\|_\infty \geq \frac{1}{2} \|((\tilde{V}_n^{(s)}(s))_u)\|_\infty /$$

$$\sqrt{N_s} \, \|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]\|_2$$

$$\|[((\tilde{S}_{n,m}(s))_{u,v}) + ((P_{n,m})_{u,v})]^{-1}\|_2$$

(299)

$$\|((\tilde{I}_n^{(0)}(s))_u)\|_\infty \geq \frac{1}{2} \|((\tilde{V}_n(s))_u)\|_\infty /$$

$$\sqrt{N_s} \, \|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]\|_2$$

$$\cdot \|[((\tilde{S}_{n,m}(s))_{u,v}) - ((P_{n,m})_{u,v})]^{-1}\|_2 \|((\tilde{Z}_{c_{n,m}}(s))_{u,v})\|_2$$

(300)

From Equations 97, 102, and 114, the supermatrix $[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]$ is given by

$$[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]$$

$$= \begin{pmatrix} (1_{n,m}) & -e^{-j\beta\ell}(\tilde{S}_{n,m}(s))_{1,2} \\ -e^{-j\beta\ell}(\tilde{S}_{n,m}(s))_{2,1} & (1_{n,m}) \end{pmatrix}$$

(301)

The 2 norm of the above matrix is given by

$$\left\| \left[\left((1_{n,m})_{u,v} \right) - \left((\tilde{\Gamma}_{n,m}(s))_{u,v} \right) : \left((\tilde{S}_{n,m}(s))_{u,v} \right) \right] \right\|_2$$

$$= \begin{array}{l} \text{square} \\ \text{root of} \\ \text{maximum} \\ \text{eigen-} \\ \text{value of} \end{array} \begin{pmatrix} (1_{n,m}) & -e^{j\beta\ell}(\tilde{S}_{n,m}(s))_{2,1} \\ \\ -e^{j\beta\ell}(\tilde{S}_{n,m}(s))_{1,2} & (1_{n,m}) \end{pmatrix}$$

$$: \begin{pmatrix} (1_{n,m}) & -e^{-j\beta\ell}(\tilde{S}_{n,m}(s))_{1,2} \\ \\ -e^{-j\beta\ell}(\tilde{S}_{n,m}(s))_{2,1} & (1_{n,m}) \end{pmatrix}$$

$$= \begin{array}{l} \text{square root of} \\ \text{maximum eigen-} \\ \text{value of} \end{array}$$

$$\begin{pmatrix} (1_{n,m}) + ((\tilde{S}_{n,m}(s))_{2,1})^2 & -e^{-j\beta\ell}(\tilde{S}_{n,m}(s))_{1,2} - e^{j\beta\ell}(\tilde{S}_{n,m}(s))_{2,1} \\ \\ -e^{j\beta\ell}(\tilde{S}_{n,m}(s))_{1,2} - e^{-j\beta\ell}(\tilde{S}_{n,m}(s))_{2,1} & (1_{n,m}) + ((\tilde{S}_{n,m}(s))_{1,2})^2 \end{pmatrix}$$

$$(302)$$

The 2 norm of the supermatrix $\left[\left((1_{n,m})_{u,v} \right) - \left((\tilde{\Gamma}_{n,m}(s))_{u,v} \right) : \left((\tilde{S}_{n,m}(s))_{u,v} \right) \right]^{-1}$ is equal to the inverse of the square root of the minimum eigenvalue of the supermatrix $\left[\left((1_{n,m})_{u,v} \right) - \left((\tilde{\Gamma}_{n,m}(s))_{u,v} \right) : \left((\tilde{S}_{n,m}(s))_{u,v} \right) \right]^{\dagger} : \left[\left((1_{n,m})_{u,v} \right) - \left((\tilde{\Gamma}_{n,m}(s))_{u,v} \right) : \left((\tilde{S}_{n,m}(s))_{u,v} \right) \right]$. The supermatrix in Equation 301 and its 2 norm at 14.375 MHz for the first three examples are obtained as

a.   Example 1--50-Ω termination

$$[((\tilde{1}_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]^{+}$$

$$: [((\tilde{1}_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]$$

$$= \begin{pmatrix} 1.2873 & 0.24155 & 0.24265 & -0.84035 & -0.42694 & -0.42887 \\ 0.24155 & 1.2873 & 0.24265 & -0.42694 & -0.84035 & -0.42887 \\ 0.24265 & 0.24265 & 1.2895 & -0.42887 & -0.42887 & -0.84422 \\ -0.84035 & -0.42694 & -0.42887 & 1.2873 & 0.24155 & 0.24265 \\ -0.42694 & -0.84035 & -0.42887 & 0.24155 & 1.2873 & 0.24265 \\ -0.42887 & -0.42887 & -0.84422 & 0.24265 & 0.24265 & 1.2895 \end{pmatrix}$$

$$(303)$$

From Equations 303 and A11, one obtains

$$\|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]\|_{2} = 1.8630 \qquad (304)$$

and

$$\|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]^{-1}\|_{2} = 3.6626 \qquad (305)$$

b.   Example 2--200-Ω termination

$$[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]^{\dagger}$$

$$: [((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]$$

$$= \begin{pmatrix} 1.2471 & 0.05188 & 0.05144 & 0.19048 & -0.66456 & -0.66649 \\ 0.05188 & 1.2471 & 0.05144 & -0.66456 & 0.19048 & -0.66649 \\ 0.05144 & 0.05144 & 1.2473 & -0.66649 & -0.66649 & 0.18604 \\ 0.19048 & -0.66456 & -0.66649 & 1.2471 & 0.05188 & 0.05144 \\ -0.66656 & 0.19048 & -0.66649 & 0.05188 & 1.2471 & 0.05144 \\ -0.66649 & -0.66649 & 0.18604 & 0.05144 & 0.05144 & 1.2473 \end{pmatrix}$$

$$(306)$$

From Equations 306 and A11, one obtains

$$\|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]\|_2 = 1.5788 \qquad (307)$$

$$\|[((1_{n,m})_{u,v}) - ((\tilde{S}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]^{-1}\|_2 = 2.2081 \qquad (308)$$

c. Example 3--50-Ω and 200-Ω terminations

$$[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]^{\dagger}$$

$$: [((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]$$

$$= \begin{pmatrix} 1.2471+j0. & 0.0512+j0. & 0.0514+j0. & -0.3249-j0.1381 & -0.5457+j0.0318 & -0.5477+j0.0318 \\ 0.0512+j0. & 1.2471+j0. & 0.0514+j0. & -0.5457+j0.0318 & -0.3249-j0.1381 & -0.5477+j0.0318 \\ 0.0514+j0. & 0.0514+j0. & 1.2473+j0. & -0.5477+j0.0318 & -0.5477+j0.0318 & -0.3291-j0.1380 \\ -0.3249+j0.1381 & -0.5457-j0.0318 & -0.5477-j0.0318 & 1.2873+j0. & 0.2416+j0. & 0.2426+j0. \\ -0.5457-j0.0318 & -0.3249+j0.1381 & -0.5477-j0.0318 & 0.2416+j0. & 1.2873+j0. & 0.2426+j0. \\ -0.5477-j0.0318 & -0.5477-j0.0318 & -0.3291+j0.1380 & 0.2426+j0. & 0.2426+j0. & 1.2895+j0. \end{pmatrix} \quad (309)$$

116

From Equations 309 and A11, one obtains

$$\|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]\|_2 = 1.7318 \qquad (310)$$

$$\|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]^{-1}\|_2 = 2.8479 \qquad (311)$$

The 2 norm of the supermatrices $[((\tilde{S}_{n,m}(s))_{u,v}) + ((P_{n,m})_{u,v})]$ and $[((\tilde{S}_{n,m}(s))_{u,v}) - ((P_{n,m})_{u,v})]$ for the three cases are obtained as

a. $\quad$ <u>Example 1--50-Ω termination</u>

$$\|[((\tilde{S}_{n,m}(s))_{u,v}) + ((P_{n,m})_{u,v})]\|_2 = 1.879$$

$$\|[((\tilde{S}_{n,m}(s))_{u,v}) + ((P_{n,m})_{u,v})]^{-1}\|_2 = 0.8327$$

$$\|[((\tilde{S}_{n,m}(s))_{u,v}) - ((P_{n,m})_{u,v})]\|_2 = 0.786 \qquad (312)$$

$$\|[((\tilde{S}_{n,m}(s))_{u,v}) - ((P_{n,m})_{u,v})]^{-1}\|_2 = 8.264$$

b. $\quad$ <u>Example 2--200-Ω termination</u>

$$\|[((\tilde{S}_{n,m}(s))_{u,v}) + ((P_{n,m})_{u,v})]\|_2 = 1.5915$$

$$\|[((\tilde{S}_{n,m}(s))_{u,v}) + ((P_{n,m})_{u,v})]^{-1}\|_2 = 1.7940$$

$$\|[((\tilde{S}_{n,m}(s))_{u,v}) - ((P_{n,m})_{u,v})]\|_2 = 1.4426 \qquad (313)$$

$$\|[((\tilde{S}_{n,m}(s))_{u,v}) - ((P_{n,m})_{u,v})]^{-1}\|_2 = 2.4479$$

c.   Example 3--50-Ω and 200-Ω terminations

$$\| [((\tilde{S}_{n,m}(s))_{u,v}) + ((P_{n,m})_{u,v})] \|_2 = 1.879$$

$$\| [((\tilde{S}_{n,m}(s))_{u,v}) + ((P_{n,m})_{u,v})]^{-1} \|_2 = 1.794$$

$$\| [((\tilde{S}_{n,m}(s))_{u,v}) - ((P_{n,m})_{u,v})] \|_2 = 1.4426 \qquad (314)$$

$$\| [((\tilde{S}_{n,m}(s))_{u,v}) - ((P_{n,m})_{u,v})]^{-1} \|_2 = 8.2874$$

Substituting Equations 304, 305, 307, 308, 310, and 311 through 314 into Equations 297 through 300, the upper and lower bounds for voltage and current for the three examples are obtained as:

a.   Example 1--50-Ω termination

$$\| ((\tilde{V}_n^{(0)}(s))_u) \|_\infty \leq 8.429 \text{ V}$$

$$\| ((\tilde{I}_n^{(0)}(s))_u) \|_\infty \leq 0.0457 \text{ A}$$

$$\qquad (315)$$

$$\| ((\tilde{V}_n^{(0)}(s))_u) \|_\infty \geq 0.133 \text{ V}$$

$$\| ((\tilde{I}_n^{(0)}(s))_u) \|_\infty \geq 0.17 \times 10^{-4} \text{ A}$$

b.   Example 2--200-Ω termination

$$\| ((\tilde{V}_n^{(0)}(s))_u) \|_\infty \leq 4.303 \text{ V}$$

$$\| ((\tilde{I}_n^{(0)}(s))_u) \|_\infty \leq 0.0505 \text{ A}$$

$$\qquad (316)$$

$$\| ((\tilde{V}_n^{(0)}(s))_u) \|_\infty \geq 0.0721 \text{ V}$$

$$\| ((\tilde{I}_n^{(0)}(s))_u) \|_\infty \geq 0.68 \times 10^{-4} \text{ A}$$

c. <u>Example 3--50-Ω and 200-Ω terminations</u>

$$\left|\left|\left(\left(\tilde{v}_n^{(0)}(s)\right)_u\right)\right|\right|_\infty \leq 6.5538 \text{ V}$$

$$\left|\left|\left(\left(\tilde{I}_n^{(0)}(s)\right)_u\right)\right|\right|_\infty \leq 0.0651 \text{ A}$$

$$\left|\left|\left(\left(\tilde{v}_n^{(0)}(s)\right)_u\right)\right|\right|_\infty \geq 0.0657 \text{ V}$$

(317)

$$\left|\left|\left(\left(\tilde{I}_n^{(0)}(s)\right)_u\right)\right|\right|_\infty \geq 0.178 \times 10^{-4} \text{ A}$$

5. SUMMARY OF RESULTS

Since this section presents several examples and various ways of calculating bounds, they are useful summarized in tabular form, as presented in Tables 1 and 2.

TABLE 1

	Calculated Values Using QV7TA Code		$\|((\tilde{v}_n^{(0)}(s))_u)\|_\infty$		$\|((\tilde{I}_n^{(0)}(s))_u)\|_\infty$	
	$\|(\tilde{v}_n^{(0)}(s))_u\|_{max}$	$\|(\tilde{I}_n^{(0)}(s))_u\|_{max}$	Upper Bound Using Eq. 253	Lower Bound Using Eq. 255	Upper Bound Using Eq. 254	Lower Bound Using Eq. 256
Example 1	0.5 V	0.01 A	19.0189 V	0.01314 V	0.2463 A	1.7×10^{-5} A
Example 2	0.5 V	2.5×10^{-3} A	4.7715 V	0.0524 V	0.0618 A	6.7×10^{-5} A
Example 3	0.8 V	4.0×10^{-3} A	19.0189 V	0.01314 V	0.2463 A	1.7×10^{-5} A
Example 4	0.5 V	6.465×10^{-4} A	1.2247 V	0.2041 V	0.0158 A	2.6×10^{-4} A

TABLE 2

	From Equations 283-286				From Equations 297-300			
	$\|((\tilde{v}_n^{(0)}(s))_u)\|_\infty$		$\|((\tilde{i}_n^{(0)}(s))_u)\|_\infty$		$\|((\tilde{v}_n^{(0)}(s))_u)\|_\infty$		$\|((\tilde{i}_n^{(0)}(s))_u)\|_\infty$	
	Upper Bound (V)	Lower Bound (V)	Upper Bound (A)	Lower Bound (A)	Upper Bound (V)	Lower Bound (V)	Upper Bound (A)	Lower Bound (A)
Example 1	7.833	0.0319	0.1319	4.1×10^{-5}	8.429	0.133	0.0457	1.7×10^{-5}
Example 2	4.208	0.0594	0.0709	7.6×10^{-5}	4.303	0.0721	0.0505	6.8×10^{-5}
Example 3	7.833	0.0319	0.1319	4.1×10^{-5}	6.5538	0.0657	0.0651	1.78×10^{-5}
Example 4	0.5	0.5	8.4×10^{-3}	6.39×10^{-4}	1.2247	0.2041	0.0158	2.6×10^{-4}

VII. CONCLUSIONS

This report has developed a formulation for the computation of upper and lower bounds on signals at terminations of a multiconductor cable network. The BLT equation expresses the characteristics of a multiconductor transmission line network in a single supermatrix notation. The upper and lower bounds on signals are obtained by using norms of vectors, matrices, supervectors, and supermatrices. Various norms and their properties for vectors, matrices, supervectors, and supermatrices are discussed.

Having developed the general formulation for the computation of upper and lower bounds on signals at terminations of a general multiconductor cable network, two special cases are considered: (1) a uniform section of a multiconductor transmission line, and (2) a multiconductor transmission line with a branch. For these two cases, scattering and propagation supermatrices are derived and their properties are discussed. The norm of the scattering supermatrix can be estimated for passive terminations. Expressions are derived for upper and lower bounds on signals for these two cases.

The upper and lower bounds are computed for a uniform three-conductor transmission line terminated at both ends in diagonal, resistive loads for several load configurations. A comparison between the calculated values and these bounds indicate that load values play an important role in determining the tightness of these bounds. In some cases, the tightness of the bounds can be improved by using ∞ norms of matrices. Further studies are required to improve upon the tightness of these bounds.

REFERENCES

1. Baum, C. E., "Electromagnetic Topology: A Formal Approach to the Analysis and Design of Complex Electronic Systems," AFWL EMP Interaction Note 400, Air Force Weapons Laboratory, Kirtland AFB, NM, September 1980.

2. Baum, C. E., T. K. Liu, and F. M. Tesche, "On the Analysis of General Multiconductor Line Networks," AFWL EMP Interaction Note 350, Air Force Weapons Laboratory, Kirtland AFB, NM, November 1978.

3. Baum, C. E., "Norms and Eigenvector Norms," Mathematics Note 63, Air Force Weapons Laboratory, Kirtland AFB, NM, November 1979.

4. Davis, W. A., and M. K. Sistanizadeh, "Bounding Signal Levels at Terminations of a Multiconductor Line Behind an Aperture," Air Force Weapons Laboratory Interaction Note 412, Kirtland AFB, NM, June 1981.

5. Tesche, F. M., and T. K. Liu, "User Manual and Code Description for QV7TA: A General Multiconductor Transmission-Line Analysis Code," AFWL Interaction Application Memo 26, Air Force Weapons Laboratory, Kirtland AFB, NM, August 1978.

6. Ralston, A., and P. Rabinowitz, _A First Course in Numerical Analysis_, Second Edition, McGraw Hill, 1978.

7. Forsythe, G., and C. B. Moler, _Computer Solution of Linear Algebraic Systems_, Prentice Hall, 1967.

8. Ogata, K., _State Space Analysis of Control Systems_, Prentice-Hall, Inc., 1967.

9. Brauer, F., and J. A. Nohel, _Qualitative Theory of Ordinary Differential Equations_, W. A. Benjamin, Inc., NY, 1969.

10. Paul, C. R., "Applications of Multiconductor Transmission Line Theory to the Prediction of Cable Coupling, Volume I," Final Technical Report, RADC-TR-76-101, Griffiss AFB, Rome Air Development Center, Rome, NY, April 1976. Also published as AFWL EMP Interaction Note 370.

11. Baum, C. E., T. K. Liu, F. M. Tesche, and S. K. Chang, "Numerical Results for Multiconductor Transmission-Line Networks," AFWL EMP Interaction Note 322, Air Force Weapons Laboratory, Kirtland AFB, NM, September 1977.

12. Agrawal, A. K., H. M. Fowles, L. D. Scott, and S. H. Gurbaxani, "Time-Domain Analysis of Multiconductor Transmission Lines with Branches in Inhomogeneous Media," Air Force Weapons Laboratory Interaction Note 333, Kirtland AFB, NM, February 1978.

13. Collin, R. E., _Foundations for Microwave Engineering_, McGraw Hill, 1966.

14. Agrawal, A. K., H. M. Fowles, and L. D. Scott, "Experimental Characterization of Multiconductor Transmission Lines in Inhomogeneous Media Using Time-Domain Techniques, Air Force Weapons Laboratory Interaction Note 332, Kirtland AFB, NM, February 1978.

15. Agrawal, A. K., "CAPCODE User's Manual," AMRC-R-140, Mission Research Corporation, Albuquerque, NM, May 1978.

APPENDIX A

NORMS OF VECTORS AND MATRICES

This appendix reviews norms of vectors, supervectors, matrices, and supermatrices. Of special interest are the norms of vectors and matrices needed to establish lower and upper bounds on the combined voltage waves and the voltages and currents in the BLT equations, derived in Section II.

A.1 VECTOR NORMS

The norm of a vector (a_n) is denoted by $\|(a_n)\|$ and it satisfies the following properties Refs. (3,6):

$$\|(a_n)\| \geq 0 \text{ with } \|(a_n)\| = 0 \text{ iff } (a_n) = (0_n)$$

$$\|\alpha(a_n)\| = |\alpha| \, \|(a_n)\|$$

$$\|(a_n) + (b_n)\| \leq \|(a_n)\| + \|(b_n)\| \tag{A1}$$

$$\|(a_n)\| \text{ depends continuously on } (a_n)$$

where

$(a_n), (b_n)$ are N-component complex vectors

α is a complex number

$|\alpha| \equiv$ magnitude of α

A common type of vector norm is referred to as the p norm defined by

$$\|(a_n)\|_p \equiv \left\{ \sum_{n=1}^{N} |a_n|^p \right\}^{1/p} \qquad \text{for any } p \geq 1 \tag{A2}$$

This has important special cases

$$\|(a_n)\|_1 \equiv \sum_{n=1}^{N} |a_n|$$

$$\|(a_n)\|_2 = \left\{ \sum_{n=1}^{N} |a_n|^2 \right\}^{1/2} \equiv \{(a_n) \cdot (a_n)^*\}^{1/2} \equiv |(a_n)| \tag{A3}$$

$$\|(a_n)\|_\infty \equiv \max_{1 \le n \le N} |a_n|$$

The 2 norm is then the euclidean norm or magnitude. The ∞ norm or maximum norm represents the magnitude of the maximum component of the vector. The 1 norm represents the sum of the magnitudes of the components of the vector.

From Equation A3, one can write

$$\|(a_n)\|_1 \ge \|(a_n)\|_2 \ge \|(a_n)\|_\infty$$

or, in general

$$\|(a_n)\|_p \ge \|(a_n)\|_q \quad p \le q \tag{A4}$$

From Equation A3, one can also write the following relations between, 1, 2, and ∞ norms

$$\|(a_n)\|_1 \le N \|(a_n)\|_\infty$$

$$\|(a_n)\|_2 \le \sqrt{N} \|(a_n)\|_\infty \tag{A5}$$

$$\|(a_n)\|_1 \le \sqrt{N} \|(a_n)\|_2$$

A.2 MATRIX NORMS

Norms can also be defined for matrices. The norm of a matrix $(A_{n,m})$ is denoted by $\|(A_{n,m})\|$ and satisfies the following properties:

$$\|(A_{n,m})\| \ge 0 \text{ with } \|(A_{n,m})\| = 0 \text{ iff } (A_{n,m}) = (0_{n,m})$$

$$\|\alpha(A_{n,m})\| = |\alpha| \|(A_{n,m})\|$$

126

$$\| (A_{n,m}) + (B_{n,m}) \| \leq \| (A_{n,m}) \| + \| (B_{n,m}) \| \tag{A6}$$

$$\| (A_{n,m}) \cdot (B_{n,m}) \| \leq \| (A_{n,m}) \| \; \| (B_{n,m}) \|$$

For the above relations to be meaningful, we must have matrices of compatible order (Ref. 2).

It follows from Equation A6 that if $(A_{n,m})$ is a square matrix, one has

$$\| (A_{n,m})^q \| \leq \| (A_{n,m}) \|^q \tag{A7}$$

$$q \equiv \text{positive integer}$$

A common way of constructing matrix norms uses the role of matrices in relating vectors via dot multiplication as in

$$(b_n) = (A_{n,m}) \cdot (X_n)$$

$$(A_{n,m}) \equiv N \times M \text{ complex matrix}$$

$$(X_n) \equiv M\text{-component complex vector} \tag{A8}$$

$$(b_n) \equiv N\text{-component complex vector}$$

If a matrix norm is defined via

$$\| (A_{n,m}) \| = \sup_{(X_n) \neq (0_n)} \frac{\| (A_{n,m}) \cdot (X_n) \|}{\| (X_n) \|} \tag{A9}$$

$$\sup \equiv \text{supremum} \equiv \text{least upper bound}$$

which makes the matrix norm a least upper bound over all (X_n) in Equation A8. The matrix norm in Equation A8 is referred to as an associated matrix norm and can be thought of as a minimum norm consistent with the chosen vector norm. Only associated norms will be used in the rest of the discussion.

127

For 1 and ∞ vector norms, the corresponding associated matrix norms are given respectively by References 3 and 6.

$$\|(A_{n,m})\|_1 = \max_{1 \le m \le M} \sum_{n=1}^{N} |A_{n,m}| \equiv \text{maximum column magnitude sum}$$

(A10)

$$\|(A_{n,m})\|_\infty = \max_{1 \le n \le N} \sum_{m=1}^{M} |A_{n,m}| \equiv \text{maximum row magnitude sum}$$

These results apply to general complex $N \times M$ matrices.

Corresponding to the vector 2 norm, the associated matrix norm is given by

$$\|(A_{n,m})\|_2 = [\lambda_{max}((A_{n,m})^{\dagger} \cdot (A_{n,m}))]^{1/2}$$

(A11)

where $\dagger$ represents conjugate transpose. Note that all the eigenvalues of $(A_{n,m})^{\dagger} \cdot (A_{n,m})$ are nonnegative since this is a positive semidefinite matrix.

For general complex square $(N \times N)$ matrices, a spectral radius can be defined as

$$\rho((B_{n,m})) \equiv \text{spectral radius of } (B_{n,m})$$

(A12)

$$\rho((B_{n,m})) = |\lambda((B_{n,m}))|_{max}$$

where $|\lambda|_{max}$ is defined as an eigenvalue of $(B_{n,m})$ with maximum magnitude.

Having defined matrix norms, we shall now derive relations between different matrix norms.

A.3 SPECTRAL RADIUS AND ASSOCIATED MATRIX NORMS

For general complex square matrices, Reference 6 gives

$$\|(A_{n,m})\| \ge \rho((A_{n,m})) = |\lambda((A_{n,m}))|_{max}$$

(A13)

so that the spectral radius is a lower bound for all associated matrix norms (for square matrices).

Equations A13 and A11 give

$$|\lambda((A_{n,m}))|_{max} \leq \lambda_{max}\{((A_{n,m})^{\dagger} \cdot (A_{n,m}))\}^{1/2} \tag{A14}$$

If $(A_{n,m})$ is real symmetric, then

$$\rho((A_{n,m})) = \|(A_{n,m})\|_2 \tag{A15}$$

and the eigenvalues of $(A_{n,m})$ are all real, since $(A_{n,m})$ is real symmetric.

A.4   RELATIONS BETWEEN 1, 2, AND ∞ ASSOCIATED MATRIX NORMS

For an $N \times M$ matrix onecan write:

a.   <u>1 and 2 norms</u>--From Equation A9, define the 1 norm of a matrix as

$$\|(A_{n,m})\|_1 = \sup_{(X_n)} \frac{\|(A_{n,m}) \cdot (X_n)\|_1}{\|(X_n)\|_1} \tag{A16}$$

From Equation A16, one has

$$\|(A_{n,m})\|_1 \leq \frac{\|(A_{n,m}) \cdot (X_n)\|_1}{(X_n)_1}$$

Substituting Equation A5 into Equation A16 gives

$$\|(A_{n,m})\|_1 \leq \sqrt{M} \frac{\|(A_{n,m}) \cdot (X_n)\|_2}{\|(X_n)\|_1}$$

$$\leq \sqrt{M} \frac{\|(A_{n,m})\|_2 \|(X_n)\|_2}{\|(X_n)\|_1}$$

$$\leq \sqrt{M} \|(A_{n,m})\|_2 \tag{A17}$$

Similarly, from Equation A9 the 2 norm of a matrix is defined as

$$\|(A_{n,m})\|_2 = \sup_{(X_n)} \frac{\|(A_{n,m}) \cdot (X_n)\|_2}{\|(X_n)\|_2} \qquad (A18)$$

From Equation A18, one has

$$\|(A_{n,m})\|_2 \leq \frac{\|(A_{n,m}) \cdot (X_n)\|_2}{\|(X_n)\|_2} \qquad (A19)$$

Substituting Equation A5 into Equation A19 yields

$$\|(A_{n,m})\|_2 \leq \frac{\|(A_{n,m}) \cdot (X_n)\|_1}{\|(X_n)\|_2}$$

$$\leq \frac{\|(A_{n,m})\|_1 \|(X_n)\|_1}{\|(X_n)\|_2}$$

$$\leq \sqrt{M} \ \|(A_{n,m})\|_1 \qquad (A20)$$

b. <u>1 and ∞ norms</u>--From Equation A9, the ∞ norm of a matrix is defined as

$$\|(A_{n,m})\|_\infty = \sup_{(X_n)} \frac{\|(A_{n,m}) \cdot (X_n)\|_\infty}{\|(X_n)\|_\infty} \qquad (A21)$$

From Equation A21, one has

$$\|(A_{n,m})\|_\infty \leq \frac{\|(A_{n,m}) \cdot (X_n)\|_\infty}{\|(X_n)\|_\infty} \qquad (A22)$$

Substituting Equation A5 into Equation A22 gives

$$\|(A_{n,m})\|_\infty \leq \frac{\|(A_{n,m}) \cdot (X_n)\|_1}{\|(X_n)\|_\infty}$$

$$\leq \frac{\|(A_{n,m})\|_1 \|(X_n)\|_1}{\|(X_n)\|_\infty}$$

$$\leq M \ \|(A_{n,m})\|_1 \qquad (A23)$$

Similarly, from Equation A16, one has

$$\|(A_{n,m})\|_1 \le \frac{\|(A_{n,m}) \cdot (X_n)\|_1}{\|(X_n)\|_1} \tag{A24}$$

Substituting Equation A5 into Equation A24 yields

$$\|(A_{n,m})\|_1 \le M \frac{\|(A_{n,m}) \cdot (X_n)\|_\infty}{\|(X_n)\|_1}$$

$$\le M \frac{\|(A_{n,m})\|_\infty \|(X_n)\|_\infty}{\|(X_n)\|_1}$$

$$\le M \|(A_{n,m})\|_\infty \tag{A25}$$

    c.   <u>2 and ∞ norms</u>--From Equation A19, one has

$$\|(A_{n,m})\|_2 \le \frac{\|(A_{n,m}) \cdot (X_n)\|_2}{\|(X_n)\|_2} \tag{A26}$$

Substituting Equation A5 into Equation A26 gives

$$\|(A_{n,m})\|_2 \le \sqrt{M} \frac{\|(A_{n,m}) \cdot (X_n)\|_\infty}{\|(X_n)\|_2}$$

$$\le \sqrt{M} \frac{\|(A_{n,m})\|_\infty \|(X_n)\|_\infty}{\|(X_n)\|_2}$$

$$\le \sqrt{M} \|(A_{n,m})\|_\infty \tag{A27}$$

Similarly, Equation A22 gives

$$\|(A_{n,m})\|_\infty \le \frac{\|(A_{n,m}) \cdot (X_n)\|_\infty}{\|(X_n)\|_\infty} \tag{A28}$$

Substituting Equation A5 into Equation A28 yields

$$\|(A_{n,m})\|_\infty \le \frac{\|(A_{n,m}) \cdot (X_n)\|_2}{\|(X_n)\|_\infty}$$

$$\le \frac{\|(A_{n,m})\|_2 \|(X_n)\|_2}{\|(X_n)\|_\infty}$$

$$\le \sqrt{M}\|(A_{n,m})\|_2 \tag{A29}$$

The relations between 1, 2, and ∞ matrix norms can now be summarized as:

$$\frac{1}{\sqrt{M}} \|(A_{n,m})\|_2 \le \|(A_{n,m})\|_1 \le \sqrt{M} \|(A_{n,m})\|_2 \tag{A30}$$

$$\frac{1}{M} \|(A_{n,m})\|_\infty \le \|(A_{n,m})\|_1 \le M \|(A_{n,m})\|_\infty \tag{A31}$$

$$\frac{1}{\sqrt{M}} \|(A_{n,m})\|_1 \le \|(A_{n,m})\|_2 \le \sqrt{M} \|(A_{n,m})\|_1 \tag{A32}$$

$$\frac{1}{\sqrt{M}} \|(A_{n,m})\|_\infty \le \|(A_{n,m})\|_2 \le \sqrt{M} \|(A_{n,m})\|_\infty \tag{A33}$$

$$\frac{1}{\sqrt{M}} \|(A_{n,m})\|_2 \le \|(A_{n,m})\|_\infty \le \sqrt{M} \|(A_{n,m})\|_2 \tag{A34}$$

$$\frac{1}{M} \|(A_{n,m})\|_1 \le \|(A_{n,m})\|_\infty \le M \|(A_{n,m})\|_1 \tag{A35}$$

where M is the number of columns of $(A_{n,m})$.

A.5   BOUNDS ON THE NORM OF SQUARE MATRICES

The spectral radius of a square matrix $(A_{n,m})$ is bounded by (Ref. 5)

$$\rho((A_{n,m})) \le \max_n \sum_{m=1}^N |A_{n,m}| \equiv \|(A_{n,m})\|_\infty \tag{A36}$$

and the spectral radius of $(A_{n,m})^{-1}$ is such that

$$\frac{1}{\rho((A_{n,m})^{-1}} \ge \min_n \left(|A_{n,n}| - \sum_{\substack{m=1 \\ m \ne n}}^N |A_{n,m}| \right)$$

or

$$\rho((A_{n,m})^{-1} \leq \frac{1}{\min\limits_{n} \left(|A_{n,m}| - \sum\limits_{\substack{m=1 \\ m \neq n}}^{N} |A_{n,m}| \right)} \tag{A37}$$

Equation A37 gives a bound for the inverse of a square matrix. The norm of a square matrix is also bounded by the following inequality (Ref. 7):

$$\max_{n,m} |A_{n,m}| \leq \|(A_{n,m})\| \leq N \max_{n,m} |A_{n,m}| \tag{A38}$$

A.6 NORM OF DIAGONAL MATRICES

In dealing with electronic systems, one often encounters matrices which are diagonal. The norms of diagonal matrices are relatively simple to evaluate. For a diagonal matrix, the associated norm is defined as

$$\|(A_{n,m})\| = \sup_{(X_n)} \frac{\|(A_{n,m}) \cdot (X_n)\|}{\|(X_n)\|}$$

$$= \sup_{(X_n)} \frac{\|(A_{n,n}X_n)\|}{\|(X_n)\|} \tag{A39}$$

Equation A39 shows that, for any p norm of the matrix, one has

$$\|(A_{n,m})\|_p = \sup_{(X_n)} \frac{\|(A_{n,n}X_n)\|_p}{\|(X_n)\|_p} \tag{A40}$$

From Equation A40, observe that

$$\max \frac{\|(A_{n,n}X_n)\|_p}{\|(X_n)\|_p} = \max_{n} |A_{n,n}| = \max_{n,m} |A_{n,m}| \tag{A41}$$

Hence,

$$\|(A_{n,m})\|_p = \max_{n,m} |A_{n,m}| \tag{A42}$$

Also from the definitions of 1, 2, and ∞ norms for matrices in Equations A10 and A11, for diagonal matrices one has

$$\|(A_{n,m})\|_1 = \|(A_{n,m})\|_2 = \|(A_{n,m})\|_\infty = \max_n |A_{n,n}| \tag{A43}$$

A.7 NORMS OF $[(1_{n,m}) + (A_{n,m})]^{-1}$ and $[(1_{n,m}) - (A_{n,m})]^{-1}$

If $\|(A_{n,m})\| < 1$, then one has (Ref. 6)

$$\|[(1_{n,m}) + (A_{n,m})]^{-1}\| \leq \frac{1}{1 - \|(A_{n,m})\|} \tag{A44}$$

To prove Equation A44, let $(B_{n,m}) = [(1_{n,m}) + (A_{n,m})]^{-1}$. Then,

$$(1_{n,m}) = [(1_{n,m}) + (A_{n,m})] \cdot (B_{n,m})$$

or

$$(B_{n,m}) = (1_{n,m}) - (A_{n,m}) \cdot (B_{n,m}) \tag{A45}$$

Taking norms of both sides and using Equation A6 gives

$$\|(B_{n,m})\| \leq \|(1_{n,m})\| + \|(A_{n,m})\| \; \|(B_{n,m})\| \tag{A46}$$

Noting that $\|(1_{n,m})\| = 1$, from Equation A46 one gets

$$\|(B_{n,m})\| \leq \frac{1}{1 - \|(A_{n,m})\|} \qquad \text{if } \|(A_{n,m})\| < 1$$

or

$$\|[(1_{n,m}) + (A_{n,m})]^{-1}\| \leq \frac{1}{1 - \|(A_{n,m})\|} \tag{A47}$$

In Equations A47, replacing $(A_{n,m})$ by $-(A_{n,m})$ gives

$$\|[(1_{n,m}) - (A_{n,m})]^{-1}\| \leq \frac{1}{1 - \|(A_{n,m})\|} \qquad \text{if } \|(A_{n,m})\| < 1 \tag{A48}$$

Note that Equation A48 has used $\|-(A_{n,m})\| = \|(A_{n,m})\|$.

Corollary 1

Let $(C_{n,m}) = e^{j\delta} (A_{n,m})$

Then
$$\| [(1_{n,m}) \pm (C_{n,m})]^{-1} \| \leq \frac{1}{1 - \|(A_{n,m})\|} \qquad (A49)$$

Since, from Equation A6,

$$\| e^{j\delta} (A_{n,m}) \| = |e^{j\delta}| \, \|(A_{n,m})\|$$

$$= \|(A_{n,m})\| \qquad (A50)$$

Corollary 2

If $(A_{n,m})$ is such that $(A_{n,m})^{-1}$ exists and $\|(A_{n,m})^{-1}\| < 1$, then

$$[(1_{n,m}) + (A_{n,m})]^{-1} = (A_{n,m})^{-1} \cdot [(1_{n,m}) + (A_{n,m})^{-1}]^{-1} \qquad (A51)$$

Taking norms of both sides and applying Equations A6 and A47, one gets

$$\| [(1_{n,m}) + (A_{n,m})]^{-1} \| \leq \frac{\|(A_{n,m})^{-1}\|}{1 - \|(A_{n,m})^{-1}\|} \qquad (A52)$$

A.8 CONDITION NUMBER OF A MATRIX

The quantity $\|(A_{n,m})\| \, \|(A_{n,m})^{-1}\|$ is defined as the condition number of $(A_{n,m})$ and is denoted as $K((A_{n,m}))$ (Ref. 6). These numbers, defined for various matrix norms, give a measure of the condition of $(A_{n,m})$ and are always greater than or equal to 1. This can be seen from the following:

$$K((A_{n,m})) = \|(A_{n,m})\| \, \|(A_{n,m})^{-1}\| \qquad (A53)$$

From the property Equation A6, one has

$$\|(A_{n,m}) \cdot (B_{n,m})\| \leq \|(A_{n,m})\| \, \|(B_{n,m})\| \qquad (A54)$$

135

Let $(B_{n,m}) = (A_{n,m})^{-1}$ (A55)

Then from Equation A54 one has

$$\|(A_{n,m})\| \; \|(A_{n,m})^{-1}\| \; \geq \; \|(1_{n,m})\|$$

and since $\|(1_{n,m})\| = 1$

one has

$$\|(A_{n,m})\| \; \|(A_{n,m})^{-1}\| \geq 1$$ (A56)

Equation A56 is valid for any associated matrix norm.

A.9 NORMS OF SUPERVECTORS

Section II introduced supervectors or divectors whose components are vectors and are defined in the form

$$((a_n)_u)$$ (A57)

with elementary vectors as

$$(a_n)_u$$

$$n = 1,2,\ldots,N_u$$ (A58)

$$u = 1,2,\ldots,N$$

The elements of supervectors are designated as

$$a_{n;u}$$ (A59)

From the definition of vector norms as defined in Equations A2 and A3, the p norm, 1 norm, 2 norm, and ∞ norm of a supervector can be defined in terms of their elements as

136

$$\|((a_n)_u)\|_p = \left\{ \sum_{u=1}^{N} \sum_{n=1}^{N_u} |a_{n;u}|^p \right\}^{1/p} \quad p \geq 1$$

$$\|((a_n)_u)\|_1 = \sum_{u=1}^{N} \sum_{n=1}^{N_u} |a_{n;u}| \qquad\qquad\qquad (A60)$$

$$\|((a_n)_u)\|_2 = \left\{ \sum_{u=1}^{N} \sum_{n=1}^{N_u} |a_{n;u}|^2 \right\}^{\frac{1}{2}}$$

$$\|((a_n)_u)\|_\infty = \max_{\substack{1 \leq n \leq N_u \\ 1 \leq u \leq N}} |a_{n;u}|$$

Note that the norms in Equation A60 satisfy properties of Equation A1.

The p norm of a supervector can be expressed in terms of the norms of its elementary vectors as

$$\|((a_n)_u)\|_p = \|(\|(a_n)_u\|_p)\|_p$$

$$= \left\{ \sum_{u=1}^{N} \|(a_n)_u\|^p \right\}^{1/p} \qquad (A61)$$

That is, the p norm of a supervector is equal to the p norm of a vector whose elements are the norms of the elementary vectors of the supervector.

From Equation A61, the 1, 2, and ∞ norms of a supervector in terms of the norms of its elementary vectors are given by

$$\|((a_n)_u)\|_1 = \sum_{u=1}^{N} \|(a_n)_u\|_1$$

$$\|((a_n)_u)\|_2 = \left\{ \sum_{u=1}^{N} \|(a_n)_u\|_2^2 \right\}^{\frac{1}{2}} \qquad (A62)$$

$$\|((a_n)_u)\|_\infty = \max_{1 \leq u \leq N} \|(a_n)_u\|_\infty$$

From Equation A62 can be written the following property for supervector (same as Eq. A4 for vectors):

$$\| ((a_n)_u) \|_1 \geq \| ((a_n)_u) \|_2 \geq \| ((a_n)_u) \|_\infty \tag{A63}$$

or, in general,

$$\| ((a_n)_u) \|_p \geq \| ((a_n)_u) \|_q \qquad p \leq q \tag{A64}$$

Similar to properties of Equation A5 for vectors, we can write the following relations for supervectors from Equation A63 as .

$$\| ((a_n)_u) \|_1 \leq N_s \| ((a_n)_u) \|_\infty$$

$$\| ((a_n)_u) \|_2 \leq \sqrt{N_s} \| ((a_n)_u) \|_\infty \tag{A65}$$

$$\| ((a_n)_u) \|_1 \leq \sqrt{N_s} \| ((a_n)_u) \|_2$$

where
$$N_s = \sum_{u=1}^{N} N_u \tag{A66}$$

A.10 NORMS OF BLOCK-DIAGONAL SUPERMATRICES

Block-diagonal supermatrices were introduced in Section II. A block-diagonal supermatrix is defined as

$$((A_{n,m})_{u,u}) = \begin{pmatrix} (A_{n,m})_{1,1} & & & \bigcirc \\ & (A_{n,m})_{2,2} & & \\ & & \ddots & \\ \bigcirc & & & (A_{n,m})_{N,N} \end{pmatrix} \tag{A67}$$

where $(A_{n,m})_{u,u}$ are square matrices of size $N_u \times N_u$. The block-diagonal super-matrix in Equation A67 may be represented in terms of the direct sum $\oplus$ as

$$((A_{n,m})_{u,u}) \equiv (A_{n,m})_{1,1} \oplus (A_{n,m})_{2,2} \oplus \cdots \oplus (A_{n,m})_{N,N}$$

$$\equiv \bigoplus_{u=1}^{N} (A_{n,m})_{u,u} \tag{A68}$$

Since $((A_{n,m})_{u,u})$ is block diagonal, its 1 and ∞ norms are given by

$$\| ((A_{n,m})_{u,u}) \|_{\substack{1 \\ \infty}} = \max_{1 \le u \le N} \| (A_{n,m})_{u,u} \|_{\substack{1 \\ \infty}} \tag{A69}$$

The 2 norm of $((A_{n,m})_{u,u})$ is given by

$$\| ((A_{n,m})_{u,u}) \|_2 = [\lambda_{max}\{((A_{n,m})_{u,u})^\dagger : ((A_{n,m})_{u,u})\}]^{\frac{1}{2}} \tag{A70}$$

Since

$$((A_{n,m})_{u,u})^\dagger : ((A_{n,m})_{u,u}) = \bigoplus_{u=1}^{N} (A_{n,m})^\dagger_{u,u} \cdot (A_{n,m})_{u,u} \tag{A71}$$

and eigenvalues of

$$((A_{n,m})_{u,u})^\dagger : ((A_{n,m})_{u,u}) = \text{eigenvalues of } \{(A_{n,m})^\dagger_{u,u} \cdot (A_{n,m})_{u,u}\}$$

$$1 \le u \le N \tag{A72}$$

Then from Equations A70 and A72, the 2 norm of $((A_{n,m})_{u,u})$ is given by

$$\| ((A_{n,m})_{u,u}) \|_2 = \max_u [\lambda_{max}\{(A_{n,m})^\dagger_{u,u} \cdot (A_{n,m})_{u,u}\}]^{\frac{1}{2}}$$

$$= \max_u \| (A_{n,m})_{u,u} \|_2$$

$$u = 1,2,\ldots,N \tag{A73}$$

Thus the 2 norm of a block-diagonal supermatrix is simply the maximum 2 norm of its block matrices on the diagonal.

A.11 NORM OF AN EXPONENTIAL FUNCTION OF A SQUARE MATRIX

If the power series

$$f(z) = \sum_{k=0}^{\infty} C_k z^k \tag{A74}$$

in a complex variable z converges everywhere, then the matrix power series

$$\sum_{k=0}^{\infty} C_k (A_{n,m})^k \tag{A75}$$

in an $N \times N$ matrix $(A_{n,m})$ converges absolutely (Ref. 8).

In the scalar case, e^z is defined by

$$e^z = 1 + z + \frac{1}{2!} z^2 + \frac{1}{3!} z^3 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!} z^k \tag{A76}$$

Since the power series

$$\sum_{k=0}^{\infty} \frac{1}{k!} z^k \tag{A77}$$

converges everywhere, the matrix power series

$$\sum_{k=0}^{\infty} \frac{1}{k!} (A_{n,m})^k \tag{A78}$$

converges absolutely for any square matrix $(A_{n,m})$. The exponential function of a matrix can thus be defined for every square matrix $(A_{n,m})$ by

$$
\begin{aligned}
e^{(A_{n,m})} &= (1_{n,m}) + (A_{n,m}) + \frac{1}{2!} (A_{n,m})^2 + \frac{1}{3!} (A_{n,m})^3 + \cdots \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} (A_{n,m})^k
\end{aligned} \tag{A79}
$$

Using Equation A6 in Equation A79 one can write

$$
\begin{aligned}
\| e^{(A_{n,m})} \| &\leq \|(1_{n,m})\| + \|(A_{n,m})\| + \frac{1}{2!} \|(A_{n,m})^2\| + \frac{1}{3!} \|(A_{n,m})^3\| + \cdots \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \|(A_{n,m})^k\|
\end{aligned} \tag{A80}
$$

Substituting Equation A7 into Equation A80 we get

$$\|e^{(A_{n,m})}\| \leq \|(1_{n,m})\| + \|(A_{n,m})\| + \frac{1}{2!}\|(A_{n,m})\|^2 + \frac{1}{3!}\|(A_{n,m})\|^3 + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!}\|(A_{n,m})\|^k = e^{\|(A_{n,m})\|} \tag{A81}$$

Therefore,

$$\|e^{(A_{n,m})}\| \leq e^{\|(A_{n,m})\|} \tag{A82}$$

Similarly, for an exponential function of $(A_{n,m})t$ we can write

$$\|e^{(A_{n,m})t}\| \leq e^{\|(A_{n,m})t\|}$$

$$= e^{\|(A_{n,m})\| \, |t|} \qquad \text{for all finite } t \tag{A83}$$

Note that, in general,

$$e^{\{(A_{n,m})+(B_{n,m})\}t} \neq e^{(A_{n,m})t} \cdot e^{(B_{n,m})t} \tag{A84}$$

unless $(A_{n,m})$ and $(B_{n,m})$ commute, that is,

$$(A_{n,m}) \cdot (B_{n,m}) = (B_{n,m}) \cdot (A_{n,m}) \tag{A85}$$

From the above discussion, one can conclude that if a function of a square matrix $(A_{n,m})$ can be expressed as a convergent infinite series as

$$f((A_{n,m})) = \sum_{k=0}^{\infty} C_k (A_{n,m})^k \tag{A86}$$

then

$$\|f((A_{n,m}))\| \leq f(\|(A_{n,m})\|) \tag{A87}$$

A.12 NORM OF FUNCTIONS INVOLVING INTEGRALS

Consider a vector expressed as an integral as

$$(a_n(z)) = \int_{z_0}^{z} (A_{n,m}(z')) \cdot (b_n(z')) \, dz' \tag{A88}$$

141

Taking norm of both sides of Equation A88 we get

$$\|(a_n(z))\| = \| \int_{z_0}^{z} (A_{n,m}(z')) \cdot (b_n(z')) \, dz' \| \qquad \text{(A89)}$$

The norm of the integral in Equation A89 satisfies the following inequality (Ref. 9):

$$\| \int_{z_0}^{z} (A_{n,m}(z')) \cdot (b_n(z')) \, dz' \| \leq \int_{z_0}^{z} \|(A_{n,m}(z'))\| \ \|(b_n(z'))\| \, dz'$$

$$\text{(A90)}$$

Substituting Equation A90 into Equation A89 gives

$$\|(a_n(z))\| \leq \int_{z_0}^{z} \|(A_{n,m}(z'))\| \ \|(b_n(z'))\| \, dz' \qquad \text{(A91)}$$

Equation A91 is an important relation which is very useful for many physical problems which often involve relations of the type in Equation A88.

A.13 NORMS OF SUPERMATRICES

Norms of supermatrices can be expressed in terms of norms of their block matrices. The 1 and ∞ norms of a supermatrix can be expressed in terms of 1 and ∞ norms of their block matrices in the following manner:

$$\max_{u,v} \|(A_{n,m})_{u,v}\|_1 \leq \|((A_{n,m})_{u,v})\|_1 \leq \max_{1 \leq v \leq M} \sum_{u=1}^{N} \|(A_{n,m})_{u,v}\|_1 \qquad \text{(A92)}$$

$$\max_{u,v} \|(A_{n,m})_{u,v}\|_\infty \leq \|((A_{n,m})_{u,v})\|_\infty \leq \max_{1 \leq u \leq N} \sum_{v=1}^{M} \|(A_{n,m})_{u,v}\|_\infty \qquad \text{(A93)}$$

where $(A_{n,m})_{u,v}$ is an elementary block matrix $(N_u \times N_v)$ of $((A_{n,m})_{u,v})$. $(A_{n,m})_{u,v}$ in general is rectangular.

The relations between 1, 2, and ∞ norms of supermatrices, similar to Equations A30 through A35, can be obtained by following the procedure in Section A.4, and the resulting relations are:

$$\frac{1}{\sqrt{N_s}}\|((A_{n,m})_{u,v})\|_2 \leq \|((A_{n,m})_{u,v})\|_1 \leq \sqrt{N_s}\|((A_{n,m})_{u,v})\|_2 \quad (A94)$$

$$\frac{1}{N_s}\|((A_{n,m})_{u,v})\|_\infty \leq \|((A_{n,m})_{u,v})\|_1 \leq N_s\|((A_{n,m})_{u,v})\|_\infty \quad (A95)$$

$$\frac{1}{\sqrt{N_s}}\|((A_{n,m})_{u,v})\|_1 \leq \|((A_{n,m})_{u,v})\|_2 \leq \sqrt{N_s}\|((A_{n,m})_{u,v})\|_1 \quad (A96)$$

$$\frac{1}{\sqrt{N_s}}\|((A_{n,m})_{u,v})\|_\infty \leq \|((A_{n,m})_{u,v})\|_2 \leq \sqrt{N_s}\|((A_{n,m})_{u,v})\|_\infty \quad (A97)$$

$$\frac{1}{\sqrt{N_s}}\|((A_{n,m})_{u,v})\|_2 \leq \|((A_{n,m})_{u,v})\|_\infty \leq \sqrt{N_s}\|((A_{n,m})_{u,v})\|_2 \quad (A98)$$

$$\frac{1}{N_s}\|((A_{n,m})_{u,v})\|_1 \leq \|((A_{n,m})_{u,v})\|_\infty \leq N_s\|((A_{n,m})_{u,v})\|_1 \quad (A99)$$

where N_s is the number of columns in the supermatrix, which is equal to the size of the supermatrix for the rectangular case.

From Equations A92 and A96 one gets

$$\frac{1}{\sqrt{N_s}}\max_{u,v}\sum_{u=1}^{N}\|(A_{n,m})_{u,v}\|_1 \leq \|((A_{n,m})_{u,v})\|_2 \leq \sqrt{N_s}\max_{v}\sum_{u=1}^{N}\|(A_{n,m})_{u,v}\|_1 \quad (A100)$$

Similarly, from Equations A93 and A97 one gets

$$\frac{1}{\sqrt{N_s}}\max_{u,v}\sum_{v=1}^{M}\|(A_{n,m})_{u,v}\|_\infty \leq \|((A_{n,m})_{u,v})\|_2 \leq \sqrt{N_s}\max_{u}\sum_{v=1}^{M}\|(A_{n,m})_{u,v}\|_\infty \quad (A101)$$

Substituting Equation A30 into Equation A100 and Equation A34 into A101, respectively, yields

$$\frac{1}{\sqrt{N_S}}\max_{u,v}\frac{1}{\sqrt{N_v}}\|(A_{n,m})_{u,v}\|_2 \leq \|((A_{n,m})_{u,v})\|_2 \leq \sqrt{N_S}\max_{v}\sum_{u=1}^{N}\sqrt{N_v}\|(A_{n,m})_{u,v}\|_2 \quad (A102)$$

$$\frac{1}{\sqrt{N_S}}\max_{u,v}\frac{1}{\sqrt{N_v}}\|(A_{n,m})_{u,v}\|_2 \leq \|((A_{n,m})_{u,v})\|_2 \leq \sqrt{N_S}\max_{v}\sum_{v=1}^{M}\sqrt{N_v}\|(A_{n,m})_{u,v}\|_2 \quad (A103)$$

where N is the number of columns in the u,v block matrix $(A_{n,m})_{u,v}$.

Equations A102 and A103 give the 2 norm of a supermatrix in terms of the 2 norms of its block matrices, and Equations 92 and 93 give the 1 and ∞ norms of a supermatrix in terms of 1 and ∞ norms of its block matrices, respectively.

144

APPENDIX B

TWO NORM OF THE SCATTERING MATRIX AT A TERMINATION
OF A UNIFORM MULTICONDUCTOR TRANSMISSION LINE

From the power conservation, the reflected power from a passive termina-
tion is always less than or equal to the incident power for physically realiz-
able systems. The power-conservation condition can be expressed in terms of
the combined voltage vectors for waves leaving and entering the termination
(for $s = j\omega$) as

$$\text{Re}[(\tilde{V}_n(s))_+ \cdot (\tilde{I}_n(s))_+^*] \leq \text{Re}[(\tilde{V}_n(s))_- \cdot (\tilde{I}_n(s))_-^*] \qquad . \qquad (B1)$$

where $(\tilde{V}_n(s))_+$ and $(\tilde{V}_n(s))_-$ are combined voltage vectors for waves leaving and
entering the termination, respectively, and $(\tilde{I}_n(s))_+$ and $(\tilde{I}_n(s))_-$ are combined
current vectors for waves leaving and entering the termination (junction). The
* represents a complex conjugate. Currents are positive into the junction.

Equation B1 can be rearranged to give

$$\text{Re}[(\tilde{I}_n(s))_+^* \cdot (\tilde{V}_n(s))_+] \leq \text{Re}[(\tilde{I}_n(s))_-^* \cdot (\tilde{V}_n(s))_-] \qquad (B2)$$

The combined voltage and current vectors are related through the characteristic-
admittance matrix of the transmission line as

$$(\tilde{I}_n(s))_+ = (\tilde{Y}_{c_{n,m}}(s)) \cdot (\tilde{V}_n(s))_+ = (\tilde{V}_n(s))_+ \cdot (\tilde{Y}_{c_{n,m}}(s))^T \qquad (B3)$$

$$(\tilde{I}_n(s))_- = (\tilde{Y}_{c_{n,m}}(s)) \cdot (\tilde{V}_n(s))_- = (\tilde{V}_n(s))_- \cdot (\tilde{Y}_{c_{n,m}}(s))^T \qquad (B4)$$

If $(\tilde{Y}_{c_{n,m}}(s))$ is real, that is, the line is lossless, then substitution of
Equations B3 and B4 into B2 yields

$$(\tilde{V}_n(s))_+ \cdot (\tilde{Y}_{c_{n,m}}(s))^T \cdot (\tilde{V}_n(s))_+^* \le (\tilde{V}_n(s))_- \cdot (\tilde{Y}_{c_{n,m}}(s))^T \cdot (\tilde{V}_n(s))_-^* \tag{B5}$$

Since the characteristic-admittance matrix is symmetric, Equation B5 reduces to

$$(\tilde{V}_n(s))_+ \cdot (\tilde{Y}_{c_{n,m}}(s)) \cdot (\tilde{V}_n(s))_+^* \le (\tilde{V}_n(s))_- \cdot (\tilde{Y}_{c_{n,m}}(s)) \cdot (\tilde{V}_n(s))_-^* \tag{B6}$$

If $(\tilde{Y}_{c_{n,m}}(s))$ is a real, symmetric matrix, it can be expressed as

$$(\tilde{Y}_{c_{n,m}}(s)) = (\tilde{A}_{n,m}(s)) \cdot (\tilde{A}_{n,m}(s)) \tag{B7}$$

in which $(\tilde{A}_{n,m}(s))$ is a real, symmetric matrix, and is defined as

$$(\tilde{A}_{n,m}(s)) = (\tilde{Y}_{c_{n,m}}(s))^{\frac{1}{2}} \tag{B8}$$

Substitution of Equation B7 into B6 yields

$$(\tilde{V}_n(s))_+ \cdot (\tilde{A}_{n,m}(s)) \cdot (\tilde{A}_{n,m}(s)) \cdot (V_n(s))_+^*$$
$$\le (\tilde{V}_n(s))_- \cdot (\tilde{A}_{n,m}(s)) \cdot (\tilde{A}_{n,m}(s)) \cdot (\tilde{V}_n(s))_-^* \tag{B9}$$

Let

$$(\tilde{B}_n(s))_+ = (\tilde{A}_{n,m}(s)) \cdot (\tilde{V}_n(s))_+ \tag{B10}$$

$$(\tilde{B}_n(s))_- = (\tilde{A}_{n,m}(s)) \cdot (\tilde{V}_n(s))_- \tag{B11}$$

Substituting Equations B10 and B11 into B9 yields

$$(\tilde{B}_n(s))_+ \cdot (\tilde{B}_n(s))_+^* \le (\tilde{B}_n(s))_- \cdot (\tilde{B}_n(s))_-^* \tag{B12}$$

The combined voltage vectors for waves leaving and entering the terminations are related through the scattering matrix $(\tilde{S}_{n,m}(s))$ of the termination as (for $s = j\omega$):

146

$$(\tilde{V}_n(s))_+ = (\tilde{S}_{n,m}(s)) \cdot (\tilde{V}_n(s))_- = (\tilde{V}_n(s))_- \cdot (\tilde{S}_{n,m}(s))^T \qquad (B13)$$

Define a new scattering matrix $(\tilde{S}'_{n,m}(s))$ so that

$$(\tilde{B}_n(s))_+ = (\tilde{S}'_{n,m}(s)) \cdot (\tilde{B}_n(s))_- = (\tilde{B}_n(s))_- \cdot (\tilde{S}'_{n,m}(s))^T \qquad (B14)$$

Substitution of Equation B14 into B12 yields

$$(\tilde{B}_n(s))_- \cdot (\tilde{S}'_{n,m}(s))^T \cdot (\tilde{S}'_{n,m}(s))^* \cdot (\tilde{B}_n(s))^*_- \leq (\tilde{B}_n(s))_- \cdot (\tilde{B}_n(s))^*_-$$
$$(B15)$$

For any eigenvector $(\tilde{X}_n(s))$ of matrix $(\tilde{S}'_{n,m}(s))^T \cdot (\tilde{S}'_{n,m}(s))^*$ with eigenvalues λ_n, one obtains

$$(\tilde{X}_n(s)) \cdot (\tilde{S}'_{n,m}(s))^T \cdot (\tilde{S}'_{n,m}(s))^* \cdot (\tilde{X}_n(s))^* = \lambda_n (\tilde{X}_n(s)) \cdot (\tilde{X}_n(s))^* \qquad (B16)$$

But according to Equation B15

$$(\tilde{X}_n(s)) \cdot (\tilde{S}'_{n,m}(s))^T \cdot (\tilde{S}_{n,m}(s)) \cdot (\tilde{X}_n(s))^* \leq (\tilde{X}_n(s)) \cdot (\tilde{X}_n(s))^* \qquad (B17)$$

Therefore,

$$\lambda_n (\tilde{X}_n(s)) \cdot (\tilde{X}_n(s))^* \leq (\tilde{X}_n(s)) \cdot (\tilde{X}_n(s))^* \qquad (B18)$$

or

$$\lambda_n \leq 1 \qquad (B19)$$

Also, $\lambda_n \geq 0$ since $(\tilde{S}'_{n,m}(s))^T \cdot (\tilde{S}_{n,m}(s))^*$ is Hermitian.

Since Equation B18 is true for any eigenvalue, one has

$$\lambda_{max}\{(\tilde{S}'_{n,m}(s))^T \cdot (\tilde{S}_{n,m}(s))^*\} \leq 1 \qquad (B20)$$

Hence

$$\|(\tilde{S}'_{n,m}(s))\|_2 \leq 1 \qquad (B21)$$

147

Substituting Equations B10 and B11 into Equation B13 gives

$$(\tilde{A}_{n,m}(s))^{-1} \cdot (\tilde{B}_n(s))_+ = (\tilde{S}_{n,m}(s)) \cdot (\tilde{A}_{n,m}(s))^{-1} \cdot (\tilde{B}_n(s))_-$$

or

$$(\tilde{B}_n(s))_+ = (\tilde{A}_{n,m}(s)) \cdot (\tilde{S}_{n,m}(s)) \cdot (\tilde{A}_{n,m}(s))^{-1} \cdot (\tilde{B}_n(s))_- \qquad (B22)$$

By comparing Equations B14 and B22, one can write

$$(\tilde{S}'_{n,m}(s)) = (\tilde{A}_{n,m}(s)) \cdot (\tilde{S}_{n,m}(s)) \cdot (\tilde{A}_{n,m}(s))^{-1} \qquad (B23)$$

Since the matrix $(\tilde{S}'_{n,m}(s))$ is obtained by performing a similarity transformation on the matrix $(\tilde{S}_{n,m}(s))$, the eigenvalues of $(\tilde{S}'_{n,m}(s))$ and $(\tilde{S}_{n,m}(s))$ are equal.

For diagonal, symmetrical loads the scattering matrix $(\tilde{S}_{n,m}(s))$ (given by Eqs. 23 and 24) is symmetric and hence

$$\|(\tilde{S}_{n,m}(s))\|_2 = \lambda_{max}(\tilde{S}_{n,m}(s)) \qquad (B24)$$

By taking the transpose of Equation B23, one obtains

$$(\tilde{S}'_{n,m}(s))^T = (\tilde{A}_{n,m}(s))^{-1} \cdot (\tilde{S}_{n,m}(s))^T \cdot (\tilde{A}_{n,m}(s)) \qquad (B25)$$

Since, for diagonal, symmetrical loads, the scattering matrix is symmetric, Equation B25 reduces to

$$(\tilde{S}'_{n,m}(s))^T = (\tilde{A}_{n,m}(s))^{-1} \cdot (\tilde{S}_{n,m}(s)) \cdot (\tilde{A}_{n,m}(s)) \qquad (B26)$$

For a multiconductor line of symmetrical configuration with diagonal, symmetrical loads, one has

$$(\tilde{Y}_{C_{n,m}}(s)) \cdot (\tilde{S}_{n,m}(s)) = (\tilde{S}_{n,m}(s)) \cdot (\tilde{Y}_{C_{n,m}}(s))$$

or

$$(\tilde{A}_{n,m}(s)) \cdot (\tilde{A}_{n,m}(s)) \cdot (\tilde{S}_{n,m}(s)) = (\tilde{S}_{n,m}(s)) \cdot (\tilde{A}_{n,m}(s)) \cdot (\tilde{A}_{n,m}(s))$$

(B27)

Equation B27 can be rearranged to give

$$(\tilde{A}_{n,m}(s)) \cdot (\tilde{S}_{n,m}(s)) \cdot (\tilde{A}_{n,m}(s))^{-1} = (\tilde{A}_{n,m}(s))^{-1} \cdot (\tilde{S}_{n,m}(s)) \cdot (\tilde{A}_{n,m}(s))$$

(B28)

By comparing Equations B23, B26, and B28, one obtains

$$(\tilde{S}'_{n,m}(s))^{T} = (\tilde{S}'_{n,m}(s))$$

(B29)

i.e., the matrix $(\tilde{S}'_{n,m}(s))$ is symmetric.

Therefore,

$$\|(\tilde{S}'_{n,m}(s))\|_2 = \|(\tilde{S}_{n,m}(s))\|_2$$

(B30)

and hence,

$$\|(\tilde{S}_{n,m}(s))\|_2 \leq 1$$

(B31)

From Equation A56, the condition number of the scattering matrix is given by

$$\|(\tilde{S}_{n,m}(s))\| \, \|(\tilde{S}_{n,m}(s))^{-1}\| \geq 1$$

or

$$\|(\tilde{S}_{n,m}(s))^{-1}\|_2 \geq \frac{1}{\|(\tilde{S}_{n,m}(s))\|_2}$$

(B32)

From Equations B31 and B32, one has

$$\|(\tilde{S}_{n,m}(s))^{-1}\|_2 \geq 1$$

(B33)

APPENDIX C

TWO NORM OF THE LOSSLESS JUNCTION SCATTERING SUPERMATRIX

From the power conservation condition, the reflected power is equal to the incident power for a lossless junction (a junction with interconnection of wires only). This power-conservation condition can be expressed as (for lossless tubes and $s = j\omega$)

$$((\tilde{V}_n(L_u,s))_u)_\nu : ((\tilde{Y}_{c_{n,m}}(s))_{u,v})^T_\nu : ((\tilde{V}_n(L_u,s))_u)^*_\nu$$

$$= ((\tilde{V}_n(0,s))_u)_\nu : ((\tilde{Y}_{c_{n,m}}(s))_{u,v})^T : ((\tilde{V}_n(L_u,s))_u)^*_\nu \tag{C1}$$

where the subscript ν is for the νth junction.

If $((\tilde{Y}_{c_{n,m}}(s))_{u,v})_\nu$ is a real, symmetric supermatrix, it can be expressed as

$$((\tilde{Y}_{c_{n,m}}(s))_{u,v})_\nu = ((\tilde{A}_{n,m}(s))_{u,v})_\nu : ((\tilde{A}_{n,m}(s))_{u,v})_\nu \tag{C2}$$

in which $((\tilde{A}_{n,m}(s))_{u,v})$ is a real, symmetric matrix, and is defined as

$$((\tilde{A}_{n,m}(s))_{u,v})_\nu = ((\tilde{Y}_{c_{n,m}}(s))_{u,v})^{\frac{1}{2}}_\nu \tag{C3}$$

Substitution of Equation C2 into C1 yields

$$((\tilde{V}_n(L_u,s))_u)_\nu : ((\tilde{A}_{n,m}(s))_{u,v})_\nu : ((\tilde{A}_{n,m}(s))_{u,v})_\nu : ((\tilde{V}_n(L_u,s))_u)^*_\nu$$

$$= ((\tilde{V}_n(0,s))_u)_\nu : ((\tilde{A}_{n,m}(s))_{u,v})_\nu : ((\tilde{A}_{n,m}(s))_{u,v})_\nu : ((\tilde{V}_n(0,s))_u)^*_\nu \tag{C4}$$

Let

$$((\tilde{B}_n(L_u,s))_u)_\nu = ((\tilde{A}_{n,m}(s))_{u,v})_\nu \colon ((\tilde{V}_n(L_u,s))_u)_\nu \qquad (C5)$$

$$((\tilde{B}_n(0,s))_u)_\nu = ((\tilde{A}_{n,m}(s))_{u,v})_\nu \colon ((\tilde{V}_n(0,s))_u)_\nu \qquad (C6)$$

Substituting Equations C5 and C6 into C4 yields

$$((\tilde{B}_n(L_u,s))_u)_\nu \colon ((\tilde{B}_n(L_u,s))_u)_\nu^* = ((\tilde{B}_n(0,s))_u)_\nu \colon ((\tilde{B}_n(0,s))_u)_\nu^* \qquad (C7)$$

The combined voltage supervectors for waves leaving and entering the junction
are related through the scattering supermatrix of the junction as

$$((\tilde{V}_n(0,s))_u)_\nu = ((\tilde{S}_{n,m}(s))_{u,v})_\nu \colon ((\tilde{V}_n(L_u,s))_u)_\nu \qquad (C8)$$

Define a scattering supermatrix $(\tilde{S}'_{n,m}(s))_{u,v})$ so that

$$(\tilde{B}_n(0,s))_u)_\nu = ((\tilde{S}'_{n,m}(s))_{u,v})_\nu \colon ((\tilde{B}_n(L_u,s))_u)_\nu \qquad (C9)$$

Substitution of Equation C9 into C7 yields

$$((\tilde{B}_n(L_u,s))_u)_\nu \colon ((\tilde{B}_n(L_u,s))_u)_\nu^*$$

$$= ((\tilde{B}_n(L_u,s))_u)_\nu \colon ((\tilde{S}'_{n,m}(s))_{u,v})_\nu^T \colon ((\tilde{S}'_{n,m}(s))_{u,v})_\nu^* \colon ((\tilde{B}_n(L_u,s))_u)_\nu^* \qquad (C10)$$

or

$$((\tilde{B}_n(L_u,s))_u)_\nu \colon [((1_{n,m})_{u,v})_\nu - ((\tilde{S}'_{n,m}(s))_{u,v})_\nu^T \colon ((\tilde{S}'_{n,m}(s))_{u,v})_\nu^*]$$

$$\colon ((\tilde{B}_n(L_u,s))_u)_\nu^* = ((0_n)_u) \qquad (C11)$$

This equation can hold only if

$$\left(\left(\tilde{S}'_{n,m}(s)\right)_{u,v}\right)^T : \left(\left(\tilde{S}'_{n,m}(s)\right)_{u,v}\right)^* = \left(\left(1_{n,m}\right)_{u,v}\right)_v$$

or

$$\left(\left(\tilde{S}'_{n,m}(s)\right)_{u,v}\right)^\dagger_v = \left(\left(\tilde{S}'_{n,m}(s)\right)_{u,v}\right)^{-1}_v \tag{C12}$$

since $\left(\left(\tilde{B}_n(L_u,s)\right)_u\right)_v$ is not zero.

The result in Equation C4 is the definition of a unitary matrix. From the definition of the 2 norm of a matrix (Eq. A11), one obtains

$$\left\|\left(\left(\tilde{S}'_{n,m}(s)\right)_{u,v}\right)_v\right\|_2 = 1 \tag{C13}$$

For lossless tubes and lossless junctions, the scattering supermatrix $\left(\left(\tilde{S}'_{n,m}(s)\right)_{u,v}\right)$ is real.

If all the multiconductor lines (tubes) connected to the junction v are identical, the scattering supermatrix $\left(\left(\tilde{S}_{n,m}(s)\right)_{u,v}\right)$ will be symmetrical (Ref. 12).

From Equations C5, C6, C8 and C9, it can be shown that

$$\left(\left(\tilde{S}'_{n,m}(s)\right)_{u,v}\right)_v = \left(\left(\tilde{A}_{n,m}(s)\right)_{u,v}\right)_v : \left(\left(\tilde{S}_{n,m}(s)\right)_{u,v}\right)_v : \left(\left(\tilde{A}_{n,m}(s)\right)_{u,v}\right)^{-1}_v \tag{C14}$$

Since the matrix $\left(\left(\tilde{S}'_{n,m}(s)\right)_{u,v}\right)_v$ is obtained by performing a similarity transformation on the matrix $\left(\left(\tilde{S}_{n,m}(s)\right)_{u,v}\right)_v$, the eigenvalues of these two matrices are equal.

Further, following the procedure in Appendix B, it can be shown that, for symmetrical configurations of multiconductor lines at the junction, the junction scattering supermatrix is symmetrical; therefore,

$$\left\|\left(\left(\tilde{S}_{n,m}(s)\right)_{u,v}\right)_v\right\|_2 = \left\|\left(\left(\tilde{S}'_{n,m}(s)\right)_{u,v}\right)_v\right\|_2 = 1 \tag{C15}$$

END

FILMED

5-84

DTIC